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The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 14
Vector algebra

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Unit 14
Vector algebra

Prepared for the Course Team
by John Berry

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Introduction

This unit is about a generalization of the real number system which will be useful when we come to do mechanics in more than one dimension. It is quite distinct from the generalization to complex numbers which we considered in *Unit 5*, although there are some points of similarity.

In three-dimensional mechanics we shall need to represent physical quantities such as mass, force, velocity, acceleration, time etc., mathematically. Most of the physical quantities that we meet can be classified into two types, called **scalars** and **vectors**. **Scalars** are quantities like length, mass, energy which require only a real number to represent their size (or magnitude). Other quantities like force, velocity, acceleration possess the two qualities of size and direction and are called **vector quantities**. You are familiar with the mathematics needed to solve problems involving scalars, which is just the mathematics of real numbers and functions. We shall extend these ideas to vector quantities.

In this unit and the unit on vector calculus we introduce methods which are an essential mathematical tool for applied mathematicians, physicists and others for solving problems in the real world. For instance you will see, in the mechanics units which follow, how elegantly Newton's laws of motion can be written down in terms of vectors. It is the aim of this unit to introduce you to: the idea of a vector; its representation (i) by an arrow on a diagram and (ii) in component form; how to add and multiply vectors; and how to differentiate a vector function of time (or of some other scalar variable).

Study guide

This unit is divided into four sections:

1. Vectors: scaling and addition
2. Components of a vector
3. Products of vectors
4. Differentiating vectors

The sections should be studied in the order in which they appear. The television programme is followed by a tape, and both are associated with Subsection 3.8.

Section 1 begins by defining a vector, and then we see how a vector can be used to represent certain familiar physical quantities. However, the unit is not about these applications but develops the algebra of vectors. Section 1 then defines the multiplication of a vector by a scalar (called **scaling**) and the addition of vectors. We see that the algebraic rules for these two operations are similar to those for real numbers. Section 2 extends the plane Cartesian coordinate system to three dimensions and gives an algebraic representation of a vector in this coordinate system. Section 3 defines two ways of multiplying vectors, called the **dot product** and the **cross product**. The television programme and tape for this unit are in this section. The television programme illustrates the use of dot and cross products by looking at geometrical applications; and the tape solves a problem posed in the television programme. Section 4 introduces vector functions and their differentiation.

The end of section exercises are there to help you to see whether you have really followed the section. As you finish each section, try the end of section exercise without referring back.

After studying this unit you should have had plenty of practice at manipulating vectors and expressions involving vectors and be ready to use them in the units that follow.

Finally, a note to those students who have previously studied *MS283*. You should be familiar with the material contained in this unit. You are recommended to work through the exercises and to revise those sections that you are weak at.

1 Vectors: scaling and addition

1.1 What is a vector?

When we did one-dimensional mechanics we had to represent various types of physical quantity, such as force, mass, velocity, energy and so on, mathematically. We found that real numbers were sufficient for representing the size or magnitude of any physical quantity, but there were some physical quantities which had a definitive direction attached to them. We used a positive real number for one direction and negative for the opposite direction.

For instance, in Section 1 of *Unit 4* we saw that the (instantaneous) velocity of a particle can be positive or negative, according to whether the particle travels in the direction of x increasing or decreasing respectively. The quantity known as *speed* was introduced, which is just the magnitude of the velocity; and to define the *velocity* of a particle fully we had to give the speed and the direction of motion.

Force is another physical quantity which requires a sign attached to its value. We represent a force of 3 newtons acting along the positive x direction by the real number $+3$, and a force of 2 newtons acting in the opposite direction by -2. (In *Units 4* and 7 we called these real numbers the *values* of the forces. The direction of a force was shown on a diagram by an arrow.)

When used in this way we can think of a (non-zero) real number x as defined by two independent pieces of information:

(i) its *modulus* $|x|$, which is a *positive* real number and is used to represent the magnitude of the force, velocity, etc.;

(ii) its *sign*, which is either + or − and is used to represent the direction of the force, velocity etc. There is also the special real number zero, which has modulus zero and which has no sign; it is neither positive nor negative.

To be able to extend our treatment of mechanics to three dimensions we need a mathematical representation for forces, velocities, etc. in any direction, not just the two opposite directions parallel to the x-axis. That is, we want a class of mathematical objects defined not just by a positive real number and a sign, but by a positive real number and a *direction in space* (or, in the case of two-dimensional problems, a direction in the plane). These mathematical objects are called vectors.

A (non-zero) **vector** consists of:

(i) a positive real number called its **modulus** or **magnitude**;

(ii) a direction in space (or in the plane).

We also define the **zero vector** to have modulus zero and no direction defined for it.

Vectors will be denoted in the text by bold type letters, e.g. $\mathbf{v}$, $\mathbf{F}$. In particular, the zero vector will be denoted by a bold zero: $\mathbf{0}$. In your written working, you should denote vectors by drawing a line under the letter e.g. $\underline{v}$, $\underline{F}$. It is important for you to practice using this notation because we shall have two types of objects:

(i) those which have no direction in space and just require a real number to represent them; these objects are called **scalars** and are denoted in the usual way, e.g. k, v;

(ii) vectors, which have a direction associated with them.

Thus if the letter v is used to represent the velocity of a particle in 3 dimensions, then it must be written by you as $\underline{v}$ (and in the text as $\mathbf{v}$).

The modulus or magnitude of the vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$ or sometimes by v, and, unless $\mathbf{v}$ is the zero vector, it is always a *positive* real number. The symbol $|\ldots|$ is the same as that used in *Unit 5* for the modulus of a complex number.

A vector can be conveniently represented in a diagram by an arrow, i.e. a straight line with an arrowhead on it. The tail of the arrow is placed at some fixed origin, the direction of the arrow is chosen to be the same as that of the vector, and its length is chosen to be equal to the magnitude of the vector (in some suitable units). In Figure 1, the shorter arrow represents a vector of magnitude 1 in the positive x direction and the longer arrow represents a vector of magnitude $2\sqrt{2}$ in a direction at 45° to the positive x direction.

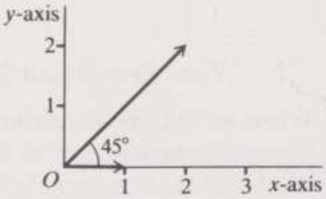


Figure 1

If we decide to denote the vectors by letters **a** and **b** respectively, then we can also put this information on the diagram by writing **a** and **b** near the arrowheads as shown in Figure 2. Thus we have an algebraic representation for a vector, and a geometric representation for a vector by an arrow.

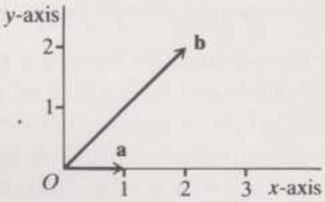


Figure 2

Exercise 1

Two vectors are defined in the following way:

(i) one vector **a** has magnitude 3 units and points in the positive y direction;

(ii) the other vector **b** has magnitude 4 units and points in the direction at 60° to the positive x direction and 30° to the positive y direction.

Represent these two vectors on a diagram by arrows.

[*Solution on p. 47*]

1.2 Examples of vector quantities

It is customary to classify physical quantities into two types: scalar quantities and vector quantities. **Scalar quantities** are those that have no direction associated with them, and can therefore be usefully represented by real numbers regardless of the number of dimensions: mass, temperature and energy are examples. **Vector quantities** are those that do have directions associated with them, and can therefore be usefully represented by real numbers only in one-dimensional problems; in two or three dimensions they must be represented by vectors instead. Here are some examples of vector quantities.

(a) Displacement

A *displacement* is a change in position, possibly as the result of some motion. Displacement is a vector quantity; the magnitude of the vector is the straight-line distance between the initial and final positions and its direction is the direction from the initial to the final position. For example, Leeds is 296 km from Bristol in the direction N 15°E; so the displacement from Bristol to Leeds (measured in km) is a vector of magnitude 269 and direction N 15° E. It can be represented in a diagram by an arrow of length 296 units, as shown in Figure 3.

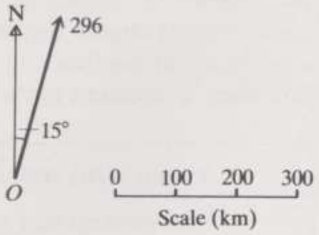


Figure 3

For any two points P and Q we define the **displacement vector** to be the vector representing the displacement from P to Q. The notation we adopt for this vector is $\overrightarrow{PQ}$. (Figure 4.)

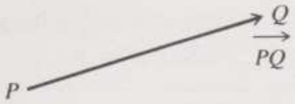


Figure 4

(b) Velocity

Another familiar physical quantity which is defined by a magnitude and a direction is *velocity*. For instance, in a weather forecast, a typical wind velocity might be 35 knots from the North-West. It is clearly not sufficient to say that 'the wind velocity is 35 knots'; the obvious question to such a statement would be 'from which direction?' The vector representing this velocity has magnitude 35 and direction from the North-West and towards the South-East (since the air is travelling in the south-easterly direction). It can be represented on a diagram as shown in Figure 5. The length of the arrow is 35 units.

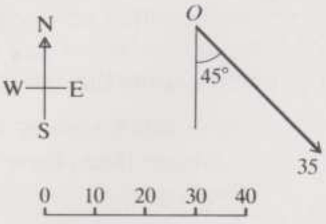


Figure 5

(c) Force

If you did *M101*, you will already be familiar (from the unit on forces) with the fact that the force acting on a given particle has magnitude and direction, and can be represented on a diagram by an arrow of suitable length in that direction. (See also the force diagrams used in *Units 4* and *7*.)

To put the directions of the vectors into these diagrams we had first to define a *reference direction.* In the examples of displacement and velocity we used the geographical references North, South, East and West. In Section 1.1 we used the x- and y-axes as reference directions and referred the vectors to the positive x direction. For many problems in *Units 4* and *7* the 'vertical' or 'horizontal' are used as reference directions because the force of gravity on a particle acts in the vertically downward direction.

The direction of a vector consists of two attributes:

(i) an *orientation*, represented by the *slope* of the line in diagrams like Figures 1 to 6,

(ii) a *sense*, represented by the *arrowhead*. For instance, the arrow representing the displacement 100 km NE in Figure 6 is a line making an angle of 45° to the North direction (the orientation) and an arrowhead pointing towards North-East as opposed to South-West (the sense).

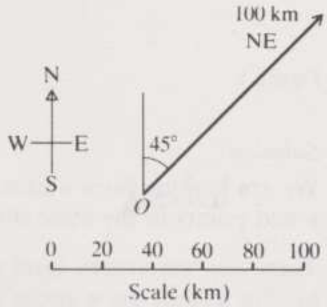


Figure 6

Exercise 2

The displacement of Birmingham from Derby is 57 km in the direction S 30° W. The displacement of Leicester from Derby is 32 km in the direction S 45° E. Draw a diagram to a suitable scale representing these two displacements by arrows.

Exercise 3

A car travelling from London along the M1 with speed 70 mph heads in the direction N 60° W at Junction 14. Represent by an arrow, drawn to a suitable scale, the velocity of the car.

Exercise 4

A stone of mass 2 kg is dropped down a well. At some instant of time the air resistance acting on the stone has magnitude 15 newtons. Draw a diagram to scale representing by arrows the two forces acting on the stone.

[*Solutions to Exercises 2–4 on p. 47*]

The solution to each of these exercises consists of an arrow representing the vector and an indication of the reference direction used. The length of the arrow represents the magnitude of the vector, the angle that the arrow makes with the reference direction is the orientation of the vector and the arrowhead gives the sense of the vector.

1.3 Some relations between vectors

In order to be able to use vectors we need to define some relations connecting different vectors. These relations are chosen because they represent useful relations between the forces, velocities etc. which the vectors may be used to represent.

(a) Equality

Two vectors are said to be **equal** if they have the same magnitude *and* the same direction.

You have seen how to represent a vector by an arrow. This definition of equality of vectors tells us that the two features needed to define a vector uniquely are its length and direction. This means that any two arrows drawn at different places on the page and which are equal in length, parallel and have the same sense can be used to represent the same vector. For instance, the two arrows in Figure 7 are each of length 2 units and point in the positive x direction. They represent two equal vectors and we write $\mathbf{b} = \mathbf{d}$.

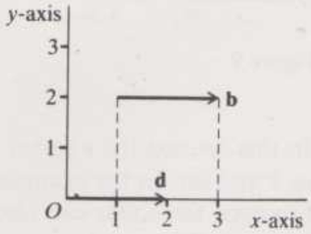


Figure 7

In other words, we do not have to draw the arrow representing a vector so that the tail starts from the origin.

Example 1

Figure 8 on the next page shows several vectors represented by arrows drawn to scale. Find the vector equal to the vector **a**.

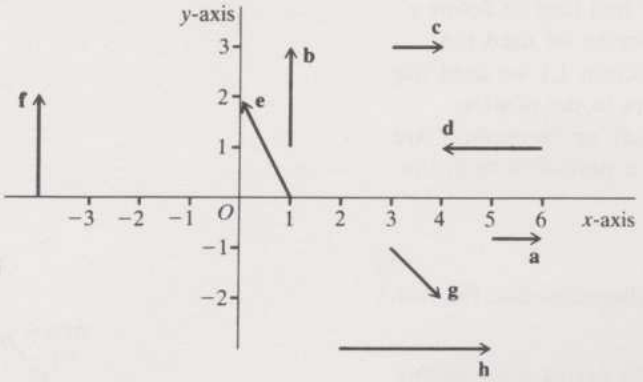


Figure 8

Solution

We are looking for a vector which is equal in length to **a** (i.e. one unit), parallel to **a** and points in the same direction (i.e. the positive x direction).

There are two arrows (and thus vectors) which point in the positive x direction besides **a**; they are **c** and **h**. (The arrow representing **d** points in the negative x direction.) The magnitudes of **c** and **h** are 1 unit and 3 units respectively. Since the magnitude of **a** is 1 unit, we must have

$$\mathbf{c} = \mathbf{a}.$$

You should note that **a** and **c** are drawn at different places in the plane. However this does not matter; **a** and **c** are equal in magnitude and have the same direction, so they are equal vectors.

Exercise 5

Which vector in Figure 8 is equal to the vector **b**?

[*Solution on p. 47*]

(b) Multiplication by a positive number

If **v** is a vector and m a positive number, then the product $m\mathbf{v}$ is a vector in the same direction as **v** but with magnitude $m|\mathbf{v}|$, i.e. m times the magnitude of **v**. This multiplication is called **scaling**.

You should note that there is no dot or cross multiplication sign between the m and the **v**. The dot and cross symbols are reserved for other products to be discussed in Section 3. For example, if **v** has magnitude 4 and points in the positive x direction, then $3\mathbf{v}$ has magnitude 12 and points in the positive x direction also. This is illustrated in Figure 9.

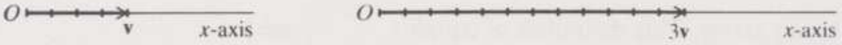


Figure 9

In this context the number m is called a **scalar** to distinguish it from vectors such as **v** and $m\mathbf{v}$. In the example above the number 3 is the scalar and **v** is the vector; however, the scalar can also be a physical scalar quantity such as time. For instance, if a particle is travelling in a straight line with constant speed and passes the origin O travelling in the positive direction at time zero, then if its displacement vector (relative to O) at time t is **s** and its velocity is **v**, we can write $\mathbf{s} = t\mathbf{v}$. This is the vector equivalent of the equation

$$x = tv$$

obtained in *Unit 4*, i.e.

distance = time × (constant) speed.

(c) Multiplication by -1

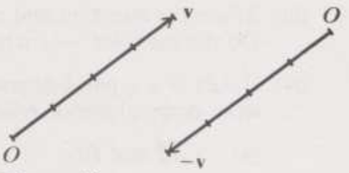


Figure 10

In one-dimensional mechanics, multiplying the value of a force by -1 would reverse its sign and would therefore be equivalent to reversing the direction of the force. Carrying this idea over to vectors in more than one dimension, we can define $-\mathbf{v}$, the vector obtained by multiplying the vector $\mathbf{v}$ and the scalar -1, to have the same magnitude as $\mathbf{v}$ but the opposite direction. This is shown in Figure 10.

(d) Multiplication by zero

What happens when we multiply a vector by zero? According to the above rule, the result should be a vector with magnitude zero. You will recall from Subsection 1.1 that there is a special vector with magnitude zero, namely the zero vector, $\mathbf{0}$. We thus have the rule

$$0\mathbf{v} = \mathbf{0}.$$

Example 2

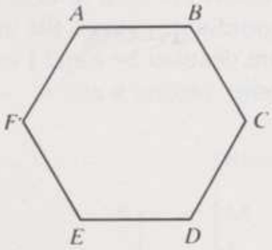


Figure 11

(i) If $\mathbf{u}$ represents the velocity of my car travelling with a speed 30 mph along a straight road due North, write down in terms of $\mathbf{u}$ the velocity of a car overtaking me travelling at 45 mph. If another car is travelling in the opposite direction to me with speed 60 mph, write down this car's velocity in terms of $\mathbf{u}$.

(ii) If $ABCDEF$ (Figure 11) is a regular hexagon and $\overrightarrow{AB}$ represents the displacement vector from A to B, write down algebraic relations connecting:

(a) $\overrightarrow{AB}$ and $\overrightarrow{ED}$,

(b) $\overrightarrow{AF}$ and $\overrightarrow{DC}$.

Solution

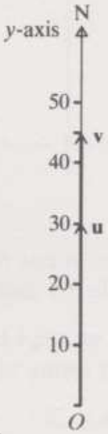


Figure 12

(i) The velocity vector $\mathbf{u}$ has magnitude 30 mph and points due North; let this direction be along the y-axis. The car overtaking me is also travelling in the positive y direction but has a velocity of magnitude 45 mph; suppose its velocity vector is denoted by $\mathbf{v}$ (Figure 12).

Then $\mathbf{v}$ is parallel to $\mathbf{u}$ and has the same sense as $\mathbf{u}$, and $|\mathbf{v}| = 1\frac{1}{2}|\mathbf{u}|$. Therefore $\mathbf{v}$ is just a scaling of $\mathbf{u}$, i.e.

$$\mathbf{v} = 1\tfrac{1}{2}\mathbf{u}.$$

Now suppose the velocity vector of the car travelling in the opposite direction is denoted by $\mathbf{w}$. Then $\mathbf{w}$ is parallel to $\mathbf{u}$ but has the *opposite* sense to $\mathbf{u}$, and $|\mathbf{w}| = 2|\mathbf{u}|$, and so we can write

$$\mathbf{w} = -2\mathbf{u},$$

where the negative sign indicates the opposing sense.

(ii) The opposite sides of a regular hexagon are parallel and all the sides have the same length. Thus the displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{ED}$ have equal magnitudes and the same directions. So we have

$$\overrightarrow{AB} = \overrightarrow{ED}.$$

The displacement vectors $\overrightarrow{AF}$ and $\overrightarrow{DC}$ have equal magnitudes but have opposite directions, thus

$$\overrightarrow{AF} = -\overrightarrow{DC}.$$

Exercise 6

(i) If $\mathbf{d}$ is the displacement vector from Bristol to Leeds, write down the displacement vectors from Leeds to Bristol and from Leeds to Leeds.

(ii) If $\mathbf{v}$ represents the velocity of a wind of 35 knots from the North–East, what vector represents:

(a) a wind of 70 knots from the North-East?,

(b) a wind of 35 knots from the South-West?,

(c) still air?

(iii) Relate the direction and magnitude of $-1.5\mathbf{v}$ to those of $\mathbf{v}$, where $\mathbf{v}$ is any given vector. Do the same for $-k\mathbf{v}$ where k is an arbitrary positive number.

(iv) If $ABCD$ is a parallelogram and $\overrightarrow{AB}$ represents the displacement vector from A to B, write down algebraic relations connecting:

(a) $\overrightarrow{AB}$ and $\overrightarrow{DC}$,

(b) $\overrightarrow{BC}$ and $\overrightarrow{DA}$.

(v) If $\mathbf{a}$ is any non-zero vector, what is the magnitude and the direction of the vector $\frac{1}{|\mathbf{a}|}\mathbf{a}$?

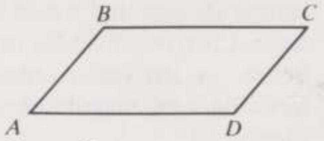


Figure 13

[Solution on p. 47]

(e) Unit vectors

The vector $\frac{1}{|\mathbf{a}|}\mathbf{a}$ in Exercise 6(v) is a special vector which has magnitude 1 and points in the direction of $\mathbf{a}$. It is called the **unit vector** in the direction of $\mathbf{a}$. Unit vectors are used to denote directions in space. For instance, in the plane Cartesian coordinate system, the unit vectors in the positive directions of the x- and y-axes are denoted by $\mathbf{i}$ and $\mathbf{j}$ respectively. Figure 14 shows these unit vectors and two other vectors $\mathbf{a}$ and $\mathbf{b}$.

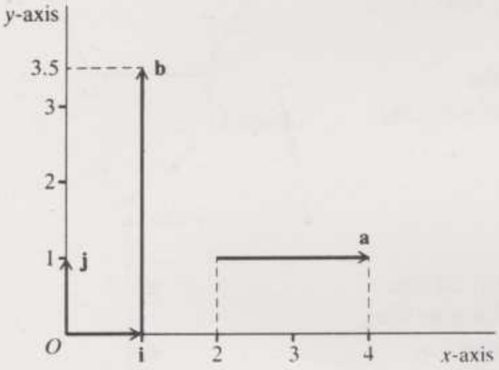


Figure 14

The vector $\mathbf{a}$ has magnitude 2 and points in the direction of x increasing; $\mathbf{b}$ has magnitude 3.5 and points in the direction of y increasing.

The unit vector $\mathbf{i}$ has magnitude 1 and points the same way as $\mathbf{a}$. Thus we can write $\mathbf{a}$ in terms of $\mathbf{i}$ by a scaling,

$$\mathbf{a} = 2\mathbf{i}.$$

Similarly we can write $\mathbf{b}$ in terms of $\mathbf{j}$,

$$\mathbf{b} = 3.5\mathbf{j}.$$

The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are called **Cartesian unit vectors**.

Exercise 7

Four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ of magnitudes 2, 2.5, 3 and 1 respectively are shown in Figure 15. The directions of the four vectors are defined by the arrows. Using scalings of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, write down $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

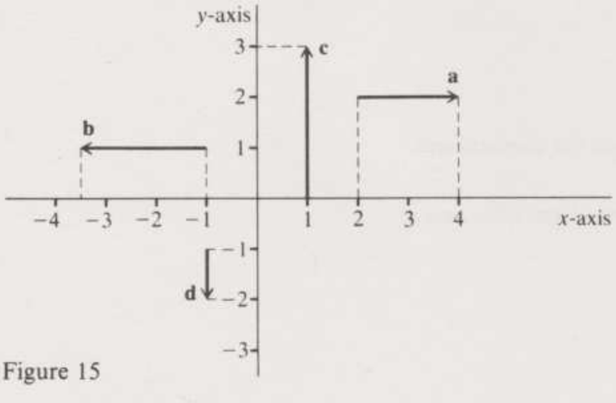


Figure 15

Exercise 8

Let the unit vectors **i** and **j** denote the directions of East and North respectively. Specify the following vectors in terms of **i** and **j**:

(i) the wind velocity of 35 km/hr towards due South,

(ii) the displacement of Bristol from London (112 miles due West),

(iii) the displacement of London from Bristol.

[*Solutions to Exercises 7 and 8 on p. 47*]

1.4 Addition of vectors

Addition of vectors is defined by a **parellelogram law** analagous to the one you have already met in *Unit 5* as a geometrical representation of the addition law for complex numbers. The rule for adding two vectors **a** and **b** is shown in the box below.

The parallelogram law of vector addition

To find the sum $\mathbf{a} + \mathbf{b}$ of any two vectors **a** and **b**:

Choose an origin O, draw the line OP_1 in the direction of **a** and with length equal to the magnitude of **a**, and draw the line OP_2 in the direction of **b** and with length equal to the magnitude of **b**. Complete the parallelogram OP_1QP_2.

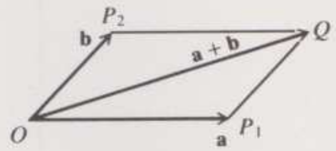


Figure 16

Then $\mathbf{a} + \mathbf{b}$ is the vector with magnitude equal to the length of OQ and with the direction of O to Q.

Exercise 9

Three vectors **a**, **b** and **c** of magnitudes 3, 2 and 4 are shown in Figure 17.

(i) Draw a rough sketch to show the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$.

(ii) Sketch the vector $-\mathbf{b}$ and draw a rough sketch to show the addition of **a** and $-\mathbf{b}$.

[*Solution on p. 47*]

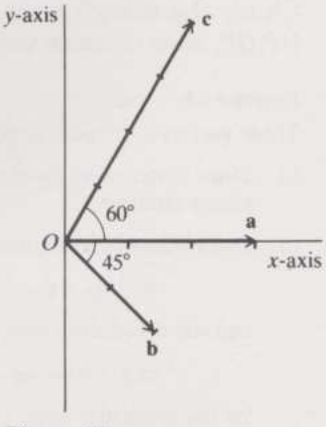


Figure 17

Exercise 9 suggests a definition of **vector subtraction**. To subtract the vector **b** from the vector **a** we *add* the vectors **a** and $-\mathbf{b}$ by the parallelogram law of vector addition; i.e. in symbols,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Exercise 10

A vector **a** has magnitude 3 units and points in the positive x direction. A vector **b** has magnitude 4 units and points in the positive y direction. Draw a diagram showing the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

[*Solution on p. 48*]

The reason for this definition of vector addition is that it enables us to use vectors to represent physical quantities which have magnitude and direction and combine in this way. For example, if you did *M101* you will have met (in the unit on forces) the parallelogram law for calculating the *resultant* of two forces (i.e. the single force which, acting on a particle, produces the same effect as the two original forces). Since the law for calculating the resultant is the same as the law for finding the sum of the vectors representing the two forces, it follows that the vector representing the resultant will simply be the sum of the vectors representing the two forces. In symbols, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are the vectors representing the two forces and **R** is the vector representing the resultant, then

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2.$$

Now let us consider displacements. Consider two consecutive journeys, one from Bristol to Leeds and the other from Leeds to Norwich. The first journey produces a displacement $\mathbf{d}_1$ and the second a displacement $\mathbf{d}_2$. The nett result is a displacement $\mathbf{d}_3$ from Bristol to Norwich.

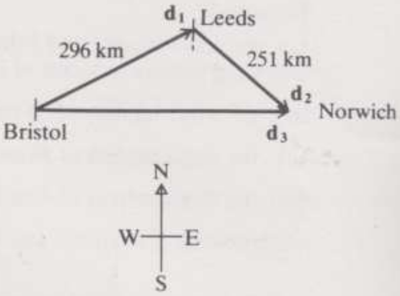


Figure 18

To relate this combination of two displacements to the parallelogram law for adding vectors, choose the origin at B, draw the arrows representing the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, and complete the parallelogram as shown. By construction, BX is equal and parallel to LN, and so the fourth corner of the parallelogram is N and the parallelogram law gives (Figure 19)

$$\mathbf{d}_1 + \mathbf{d}_2 = \mathbf{d}_3.$$

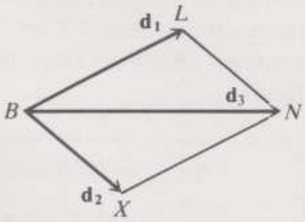


Figure 19

That is, combining two consecutive displacements is equivalent to adding the vectors representing them.

This alternative rule, the **triangle rule**, of adding displacment vectors applies to the addition of any two vectors and can be used as an alternative definition of vector addition.

The triangle law of vector addition

To find the sum $\mathbf{a} + \mathbf{b}$ of any two vectors $\mathbf{a}$ and $\mathbf{b}$:

choose an origin O, draw the line OP_1 in the direction of $\mathbf{a}$ and with length equal to the magnitude of $\mathbf{a}$, and draw the line P_1Q in the direction of $\mathbf{b}$ and with length equal to the magnitude of $\mathbf{b}$.

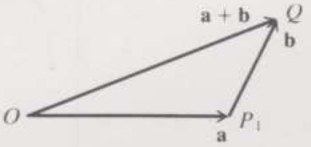


Figure 20

Then $\mathbf{a} + \mathbf{b}$ is the vector with magnitude equal to the length of OQ and with the direction from O to Q.

Clearly this triangle is the same as the lower triangle OP_1Q in the parallelogram OP_1QP_2 used to define vector addition in the parallelogram law (page 11).

Exercise 11

Three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are defined as in Exercise 9.

(i) Draw sketches to show the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} + \mathbf{c}$ and $\mathbf{c} - \mathbf{b}$ using the triangle rule of vector addition.

(ii) Draw sketches to illustrate the associative property of vector addition

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}),$$

and the distributive property of scaling over vector addition

$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b},$$

for the particular cases of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ above and the scalar $m = 2$.

[*Solution on p. 48*]

1.5 Algebraic rules for scaling and addition of vectors

In Subsections 1.3 and 1.4 we showed how to multiply a vector by a scalar and how to add vectors, i.e. what is meant by $m\mathbf{v}$ and $\mathbf{a} + \mathbf{b}$. Exercise 11 (ii) illustrated, for vectors lying in the same plane, the associative property of vector addition and the distributive property of scaling over vector addition, i.e. $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$, for the special case $m = 2$. Clearly vector addition is commutative, i.e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, because the same parallelogram is used to define each side of the equation.

These are some of the algebraic rules for manipulating vectors by addition and scaling. A complete list of these rules is given below. They apply whether or not the vectors are confined to a plane.

1. If **a** and **b** are vectors, then $\mathbf{a} + \mathbf{b}$ is a vector defined by the parallelogram rule or the triangle rule.
2. Addition is commutative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
3. Addition is associative: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.
4. If **a** is a vector and m is a scalar, then $m\mathbf{a}$ is a vector.
5. Scaling is associative in the sense that $m_1(m_2\mathbf{a}) = (m_1m_2)\mathbf{a}$.
6. Scaling is distributive in the sense that $(m_1 + m_2)\mathbf{a} = m_1\mathbf{a} + m_2\mathbf{a}$.
7. Scaling is distributive over vector addition:
$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$$
8. The zero vector behaves as expected: $0\mathbf{a} = \mathbf{0}$ and $\mathbf{0} + \mathbf{a} = \mathbf{a}$.

These rules are similar to the properties of addition of ordinary real and complex numbers, and with them we can manipulate algebraic expressions involving scalings and vector addition according to familiar rules. For example, we can simplify the expression

$$2(\mathbf{a} + \mathbf{b}) + 3(\mathbf{b} + \mathbf{c}) - 5(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

to obtain

$$8\mathbf{c} - 3\mathbf{a}.$$

Notice that these rules say nothing about the multiplication of one vector by another; we study how to multiply vectors later on. Nor have we said anything about division by a vector: in fact it is NOT possible to divide by a vector.

Exercise 12

Find the vector **x** in terms of **a** and **b** in the following vector equations:

(i) $2\mathbf{b} + 4\mathbf{x} = 7\mathbf{a}$,

(ii) $n(\mathbf{b} - \mathbf{a}) + \mathbf{x} = m(\mathbf{a} - \mathbf{b})$.

[*Solution on p. 48*]

Summary of Section 1

1. A vector is a quantity that consists of:
   - (i) a magnitude and
   - (ii) a definite direction in space.
2. Vectors are denoted in two ways:
   - (i) algebraically, by a bold type letter **a** (or an underlined letter $\underline{a}$ in written work), or by a symbol such as $\overrightarrow{AB}$;
   - (ii) on a diagram, by an arrow, the length of which is proportional to the magnitude and whose direction is the same as that of the vector.
3. Displacement, velocity and force are three examples of vector quantities that occur in mechanics.
4. (i) We can multiply a vector **a** by a number m (called here a **scalar**) to get a vector denoted by $m\mathbf{a}$. The direction of $m\mathbf{a}$ is the same as **a** if m is positive, and opposite to **a** if m is negative. The magnitude of $m\mathbf{a}$ is $|m|$ times the magnitude of **a**.
   - (ii) If $m = 0$, then $m\mathbf{a}$ is the **zero vector**, denoted by **0**, with zero magnitude and no direction in space.
   - (iii) If $m = \frac{1}{|\mathbf{a}|}$, the vector $m\mathbf{a}$ has magnitude one and is called a **unit vector**.
   - (iv) If $m = -1$, the vector $m\mathbf{a}$ is denoted by $-\mathbf{a}$.

5. The **Cartesian unit vectors i** and **j** point along the positive x- and y-axes respectively.

6. **Vector addition** is defined by the **parallelogram rule**: to add $\mathbf{b}$ to $\mathbf{a}$, draw the parallelogram with two sides parallel to $\mathbf{a}$ and equal in length to $\mathbf{a}$, and the other two sides parallel and equal in length to $\mathbf{b}$. The arrow $\overrightarrow{OR}$ in Figure 21 represents the vector $\mathbf{a} + \mathbf{b}$.

7. **Vector subtraction** is defined in terms of vector addition by

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

8. The algebraic rules for manipulating vector addition and scaling are given in the box on page 13.

9. Vectors can be added by the **triangle rule**. This is an alternative version of the parallelogram addition rule and gives the same result.

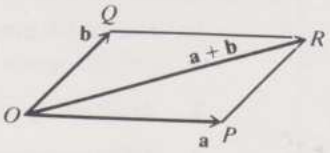


Figure 21

End of section exercises

Exercise 13

The following is a list of some physical quantities. Decide which are scalar and which are vector quantities.

area, temperature, velocity, volume, energy, force, displacement, time, acceleration.

Exercise 14

The vectors $\mathbf{a}$ and $\mathbf{b}$ are represented by the arrows shown in Figure 22. The magnitudes of $\mathbf{a}$ and $\mathbf{b}$ are 4 and 6 units respectively. Draw a sketch to show the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and $2\mathbf{a} + \frac{1}{2}\mathbf{b}$.

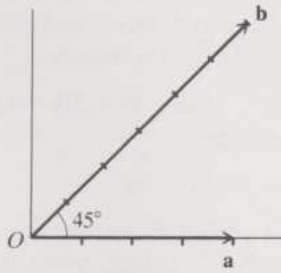


Figure 22

Exercise 15

If $\mathbf{v} = -4.7\mathbf{u}$, what can we say about the magnitude and direction of $\mathbf{v}$ in terms of the magnitude and direction of $\mathbf{u}$?

Exercise 16

If $ABCD$ is a quadrilateral and $\overrightarrow{AB}$ is the displacement vector from A to B, and similarly for $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, show that

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0}.$$

[*Solutions to Exercises 13–16 on pp. 48–49*]

2 Components of a vector

So far we have approached vectors, and the laws of vector addition and scaling, geometrically. To add vectors geometrically requires drawing scale diagrams representing the vectors by arrows. An alternative, and sometimes more convenient, approach for representing a vector is developed in this section. Using the laws of scaling and vector addition we represent a vector *algebraically*, in two dimensions by two numbers called **components** and in three dimensions by three such components.

2.1 Vectors in two dimensions

The theory of complex numbers in *Unit 5* illustrates some of the advantages of the algebraic approach we are seeking. We can represent a complex number z by two real numbers x and y, and then to add the two complex numbers z_1 $(=x_1 + iy_1)$ and z_2 $(=x_2 + iy_2)$, we simply add the real parts x_1 and x_2 and the imaginary parts y_1 and y_2 separately. We shall be able to represent our vectors in terms of real numbers called **components**, and the operations of adding and scaling of vectors just require the addition and scaling of components.

We have already seen in Subsection 1.3 how to write some vectors as scalar multiples of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. For example, Figure 1 shows two vectors $\mathbf{a}$ and $\mathbf{b}$ of magnitude 2 and 3 units respectively, both pointing in the positive x direction.

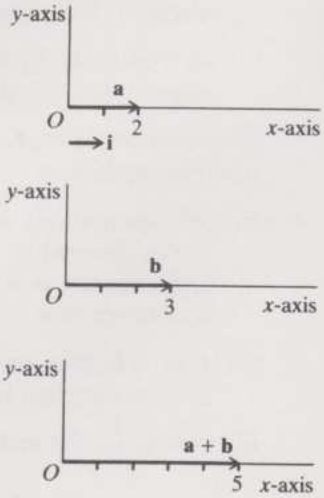


Figure 1

The sum $\mathbf{a} + \mathbf{b}$ is a vector of magnitude 5 units pointing in the positive x direction, also shown in Figure 1. The vectors **a** and **b** are both parallel to the Cartesian unit vector **i** and can be represented as scalings of **i**:

$$\mathbf{a} = 2\mathbf{i} \quad \text{and} \quad \mathbf{b} = 3\mathbf{i}.$$

Using the rules for manipulating vector addition we get

$$\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 3\mathbf{i} = 5\mathbf{i},$$

i.e. a vector in the direction of **i** and of magnitude 5 units. The example demonstrates the geometric and algebraic approaches to vector addition in one dimension.

Of course, in general our vectors do not lie parallel to either the x- or y-axes and may not even lie in the xy-plane, and so we must expand these simple ideas.

Consider the arbitrary vector **a** in the xy-plane as shown in Figure 2. The position of the end of the vector $\overrightarrow{OA}$ can be described in terms of the Cartesian coordinates x and y, i.e. the distances OB and OC respectively. The vectors $\overrightarrow{OB}$ and $\overrightarrow{OC}$ can be written as scalings of the Cartesian unit vectors **i** and **j**:

$$\overrightarrow{OB} = x\mathbf{i} \quad \text{and} \quad \overrightarrow{OC} = y\mathbf{j},$$

and the parallelogram rule for the addition of vectors allows the vector **a** to be expressed as the sum of $\overrightarrow{OB}$ and $\overrightarrow{OC}$:

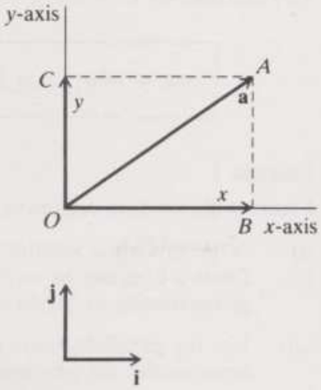


Figure 2

$$\boxed{\mathbf{a} = x\mathbf{i} + y\mathbf{j}.}$$

This is called the **component form** of **a**. The vectors $x\mathbf{i}$ and $y\mathbf{j}$ are called the **(Cartesian) component vectors** of **a**, and the numbers x and y are called the **(Cartesian) components** of **a**. Note the difference between *component vectors* (which are vectors!) and *components*, which are numbers.

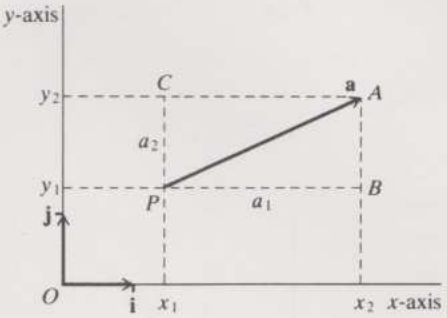
Any two vectors drawn in the plane can be added quite simply by adding components. For example, if $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j}$, then

$$\mathbf{a} + \mathbf{b} = (2\mathbf{i} + 3\mathbf{j}) + (3\mathbf{i} + \mathbf{j}) = 5\mathbf{i} + 4\mathbf{j}.$$

A subscript notation for the components is also useful and we write

$$\boxed{\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.}$$

The components a_1 and a_2 are shown in Figure 3, in which PB and PC are parallel to the x- and y-axes respectively.



$$\boxed{\begin{aligned} a_1 &= x_2 - x_1 \\ a_2 &= y_2 - y_1 \end{aligned}}$$

Figure 3

The magnitude of the vector **a** is just the length of line PA, and in terms of a_1 and a_2 this is

$$\boxed{|\mathbf{a}| = \sqrt{{a_1}^2 + {a_2}^2}.}$$

As in the above example, the associative rule for vector addition and the distributive rule for scaling allow us to add vectors by adding components: if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\boxed{\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}.}$$

Also the fact that scaling is distributive over vector addition (Rule 7, Subsection 1.5) enables us to scale a vector **a** by scaling its components:

$$\boxed{m\mathbf{a} = m(a_1\mathbf{i} + a_2\mathbf{j}) = (ma_1)\mathbf{i} + (ma_2)\mathbf{j}.}$$

Exercise 1

Figure 4 shows four vectors in the *xy*-plane.

(i) Write down the vectors in Cartesian component form.

(ii) Draw a diagram to verify that the scaling $3.5\mathbf{a}$ is the same when obtained geometrically or algebraically using components.

(iii) Use the parallelogram rule to obtain the vector $\mathbf{a} + \mathbf{b}$. Verify that this vector is the same as the one obtained by adding the component forms of **a** and **b**.

(iv) Find, algebraically, the components of the vector $2\mathbf{a} + \mathbf{b} - \mathbf{c}$. Hence find the magnitude of the vector $2\mathbf{a} + \mathbf{b} - \mathbf{c}$.

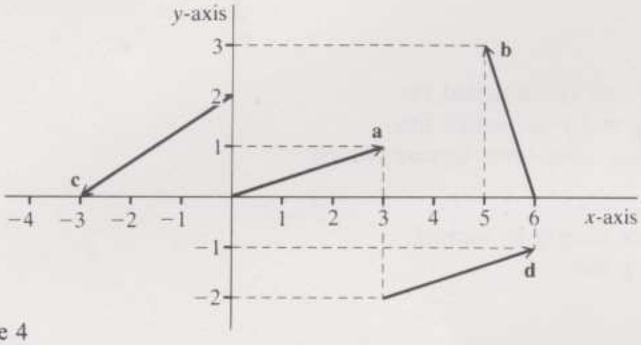


Figure 4

Exercise 2

Three vectors **a**, **b** and **c** all in the same plane are defined by specifying for each of them a length, an angle θ to the line LL' measured anticlockwise from the LL' direction, and a sense denoted by the arrowhead, as shown in Figure 5.

a: magnitude 2 and $\theta = 60°$

b: magnitude 3 and $\theta = 135°$

c: magnitude 1 and $\theta = 30°$.

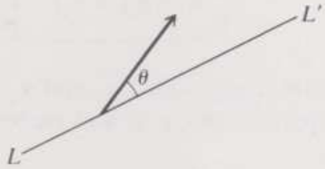


Figure 5

(i) Draw a picture illustrating the vectors **a**, **b** and **c**.

(ii) Choose unit vectors **i** and **j** along and perpendicular to LL' respectively; write down the vectors **a**, **b**, **c** and $\mathbf{a} + \mathbf{c}$ in terms of **i** and **j**.

[*Solutions to Exercises 1 and 2 on p. 49*]

This last exercise demonstrates that once suitable unit vectors **i** and **j** have been chosen, any vector **a** in the plane of **i**, **j** can be represented in component form, and then the components of **a** are given in terms of the magnitude of **a** and the angle that **a** makes with the **i** direction.

Figure 6 shows a vector **a** making an angle θ with the **i** direction. The vector **a** is given in component form by

$$\begin{aligned}\mathbf{a} &= (|\mathbf{a}|\cos\theta)\mathbf{i} + (|\mathbf{a}|\sin\theta)\mathbf{j}\\ &= |\mathbf{a}|\,(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta).\end{aligned}$$

This is similar to the polar form of a complex number.

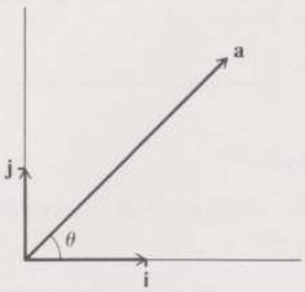


Figure 6

This expression can also be used in reverse to find the length and direction of a vector given in component form. If the vector **a** is given as usual by $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, then

$$a_1 = |\mathbf{a}|\cos\theta \quad \text{and} \quad a_2 = |\mathbf{a}|\sin\theta,$$

so that

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \text{ (as before), and}$$
$$\theta = \pm\arccos(a_1/|\mathbf{a}|),$$

where the positive sign is chosen if $a_2 \geqslant 0$ and the negative sign if $a_2 < 0$. (This is the same equation for finding θ as that given in *Unit 5* for finding the argument of a complex number.) The angle θ lies in the range $-\pi < \theta \leq \pi$, and the direction of the vector **a** is completely described by giving the angle θ.

Exercise 3

Find the magnitudes and directions of the vectors

$$\mathbf{a} = \sqrt{3}\ \mathbf{i} - \mathbf{j} \text{ and } \mathbf{b} = -3\mathbf{i} + 3\mathbf{j}.$$

[*Solution on p. 49*]

2.2 Three-dimensional Cartesian coordinates

In *Units 4*, *7* and *8* you were introduced to the motion of particles constrained to move in a straight line, and it was sufficient to choose one coordinate to define the position of the particle at any time.

Now suppose that we have a problem of the motion of a particle not constrained to move in a straight line but free to move about in space. Such a body may be constrained to move in a plane, in which case we can use vectors in two dimensions to represent the properties of the motion, such as the position of the body at any time. For example, a golf ball hit straight, without any sidespin, will travel in a vertical plane and we can define its position by two numbers, the vertical and horizontal distances from the point of projection.

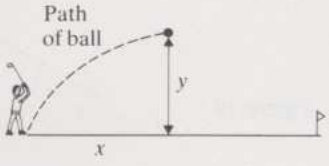


Figure 7

However, suppose that the ball is hit with sidespin so that its motion veers off from the intended direction and does not lie wholly in a vertical plane. We shall require a third number to define its position, i.e. we shall need a three-dimensional coordinate system and vectors with three components instead of two. We extend our two-dimensional Cartesian coordinate system in the following way.

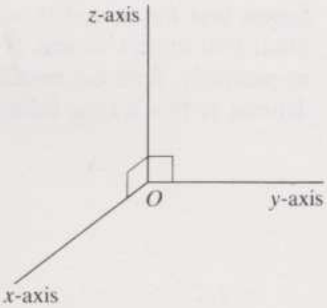


Figure 8

Through the origin O of the two-dimensional Cartesian coordinate system, draw a third axis, the **z-axis**, perpendicular to both the x- and y-axes of the two-dimensional system. This produces a coordinate system with three mutually perpendicular axes, the x-, y- and z-axes. Alternatively, it can be characterized by three planes:

the **xy-plane**, which contains the x- and y-axes and is perpendicular to the z-axis;

the **xz-plane**, analogously defined;

and the **yz-plane**, again analogously defined.

Any point P can be represented uniquely by its perpendicular distances from the xy-, xz- and yz-planes. These distances, called the (**Cartesian**) **coordinates** of P, are shown in Figure 9 on the next page. We denote the point P by the ordered triple (x, y, z), where:

$$x = PR = OA,$$
$$y = PS = OB$$
$$\text{and} \quad z = PQ = OC.$$

(In Figure 9, PQ, AQ and BQ are perpendicular to the xy-plane, xz-plane and yz-plane respectively.)

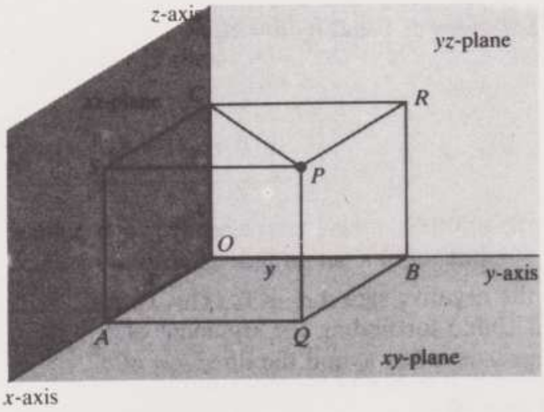


Figure 9

For example, the point (2, 3, 4) is shown in Figure 10. Positive values of z represent points above the xy-plane and negative values of z represent points below the xy-plane.

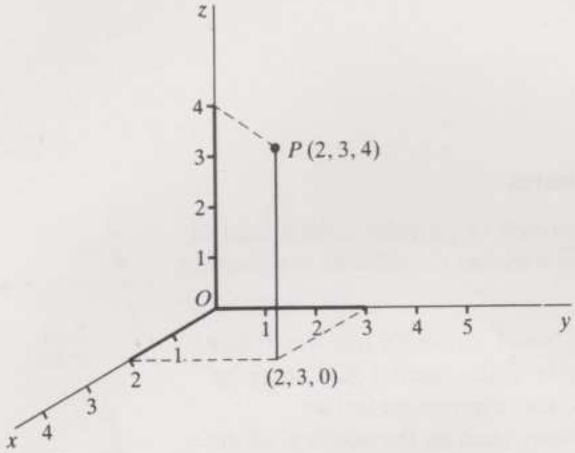


Figure 10

Clearly there are two possible ways for the positive z direction to be defined; these are shown in Figure 11.

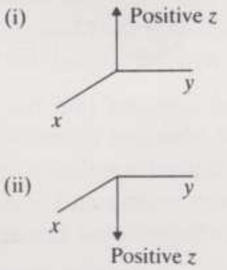


Figure 11

The usual convention for relating the positive directions of x, y and z is given by the following rule, called the **right hand rule**: The right hand is put with the middle finger, first finger and thumb placed (roughly) perpendicular to each other and the other two fingers closed. If the thumb and first finger represent the x- and y-axes respectively, then the middle finger represents the z-axis. A coordinate system defined in this way is called a **right-handed system**.

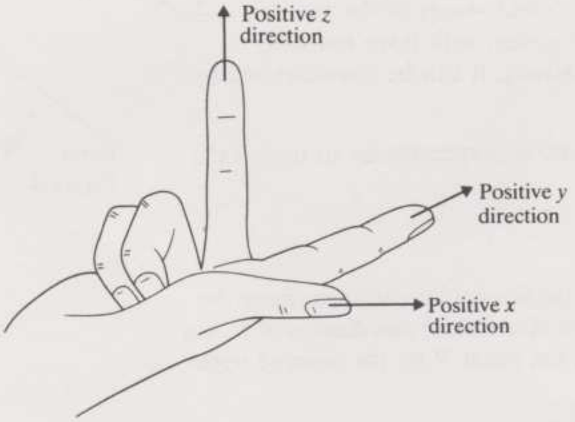


Figure 12

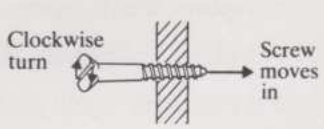


Figure 13

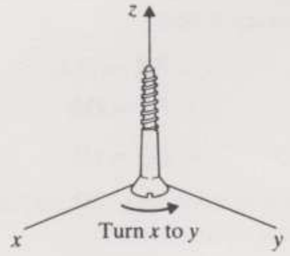
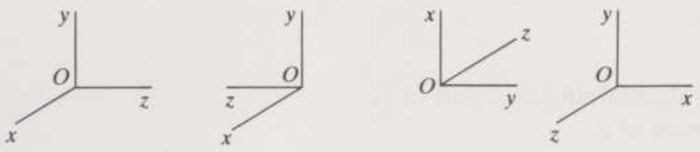


Figure 14

An alternative definition of the positive z direction is given by the **right hand screw rule**, stated as follows: Suppose we are turning a screw into a piece of wood, then a clockwise rotation makes the screw move into the wood (Figure 13). If we turn the screw from x to y, as shown in Figure 14, then the direction that the screw moves is along the positive z direction. For the rest of this unit we shall use the screw rule to characterize a right-handed system, but you should use whichever rule you find easier to apply.

Exercise 4

Decide which of the following sets of perpendicular axes define right-handed coordinate systems.



(i) (ii) (iii) (iv)

Figure 15

[*Solution on p. 49*]

2.3 Vectors in three dimensions

We can extend the algebraic representation for vectors to vectors in three dimensions. Consider Figure 16.

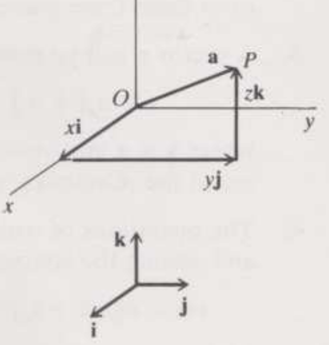


Figure 16

A third Cartesian unit vector **k** is introduced to represent the positive z direction. We now have three unit vectors **i**, **j** and **k** which are perpendicular to each other.

For any vector **a** we may write

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

or, in subscript notation

$$\boxed{\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.}$$

Since **a** is the vector from O to P, it describes the position of the point P relative to the origin O, and is therefore called the **position vector of *P* relative to *O***. When the origin has been chosen in a problem, we usually omit the words 'relative to O' and just refer to the **position vector of *P***.

As in two dimensions, the operations of vector algebra can be expressed in terms of components.

Given a scalar m and vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then

$$\boxed{\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}}$$

i.e. corresponding components add, and

$$\boxed{m\mathbf{a} = (ma_1)\mathbf{i} + (ma_2)\mathbf{j} + (ma_3)\mathbf{k}}$$

i.e. each component of **a** is scaled by m.

Finally, the magnitude of the vector **a** is given in terms of the components a_1, a_2 and a_3 by

$$\boxed{|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.}$$

Exercise 5

Given vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{k}$:

(i) evaluate $\mathbf{d} = 2\mathbf{a} - 3\mathbf{b}$ and $\mathbf{e} = \mathbf{a} - 2\mathbf{b} + 4\mathbf{c}$;

(ii) find the magnitudes of the vectors **d** and **e**;

(iii) evaluate $|\mathbf{a}|$ and write down a unit vector in the direction of $\mathbf{a}$;

(iv) find the components of $\mathbf{x}$ such that $\mathbf{a} + \mathbf{x} = \mathbf{b}$.

[*Solution on p. 49*]

Summary of Section 2

1. A vector **a** in two dimensions can be represented algebraically by a pair of numbers a_1 and a_2 called the **(Cartesian) components** of **a**,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

2. A right-handed three-dimensional Cartesian coordinate system is defined in terms of three perpendicular axes, the **x-,y- and z-axes**, whose positive directions can be related by the **right hand rule** or the **screw rule**. These axes define three planes, the **xy-,xz- and yz-planes**. Any point in space can be represented in this coordinate system by giving the perpendicular distances from these three planes. The position of the point is denoted by (x, y, z).

3. A vector **a** can be represented algebraically in the form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

where **k** is a unit vector along the positive z-axis. The numbers a_1, a_2, a_3 are called the **(Cartesian) components** of **a**.

4. The operations of scaling and adding vectors can be carried out by scaling and adding the components of the vectors:

$$m\mathbf{a} = m(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = (ma_1)\mathbf{i} + (ma_2)\mathbf{j} + (ma_3)\mathbf{k}$$

and

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}.\end{aligned}$$

End of section exercise

Exercise 6

If $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = \mathbf{j} - 2\mathbf{k}$:

(i) find the magnitudes of **a** and **b** and describe the direction of **a**;

(ii) find the vectors $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} - \mathbf{b}$ and $\mathbf{c} + 2\mathbf{b} - 3\mathbf{a}$ in component form;

(iii) what is the end point of the displacement represented by the vector $2\mathbf{a} - \mathbf{b}$ if $(0, 2, 3)$ is its starting point?

[*Solution on p. 50*]

3 Products of vectors

So far we have defined the two algebraic operations of vector addition (by the parallelogram or triangle rule) and the scaling of a vector. The addition of vectors can only be usefully applied to two vectors representing the same type of physical quantity. For example, the addition of a force and a velocity has no physical meaning. However, vectors representing different types of physical quantities can be combined in operations that we call the **dot product** and the **cross product**. They are called products because in some respects they behave like 'multiplications' in the algebra of real numbers.

In this unit we shall define the operations of **dot product** and **cross product**, and give you practice at manipulating the definitions and finding the products of vectors given in component form. The reason why the products are defined as they are will become more apparent in the next two units on mechanics.

3.1 The dot product

Suppose that we are given two vectors **a** and **b**, and θ is the angle between the directions of the two vectors; then the **dot product** of the two vectors **a** and **b** is defined by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta.$$

The notation used is very important; make sure that the dot between the vectors **a** and **b** is clear. This product is pronounced '**a** dot **b**'. The dot product of two vectors is *a scalar quantity*, in other words it is a real number; $\mathbf{a}\cdot\mathbf{b}$ is the product of the three scalars $|\mathbf{a}|$, $|\mathbf{b}|$ and $\cos\theta$. So the operation of the dot product combines two vectors to define a scalar, and for this reason the dot product is also called the **scalar product**.

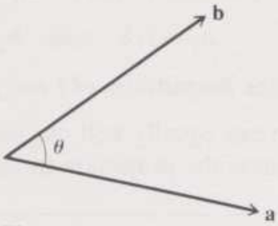


Figure 1

Exercise 1

Three vectors **a**, **b** and **c** of magnitudes 2, 4 and 1 unit respectively and lying in the same plane are represented by arrows as shown in Figure 2. The angle between the vectors **a** and **b** is 60° and between the vectors **b** and **c** is 30°. Use the definition of dot product to find the values of $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\cdot\mathbf{c}$, $\mathbf{b}\cdot\mathbf{c}$ and $\mathbf{b}\cdot\mathbf{b}$.

[*Solution on p. 50*]

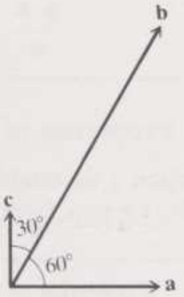


Figure 2

This exercise demonstrates two important properties of the dot product:

(i) If **a** and **b** are two vectors which are perpendicular to each other (i.e. the angle between them is 90°), then since $\cos 90^\circ = 0$ we get

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos 90^\circ = 0.$$

(ii) The dot product of a vector with itself gives the square of the magnitude of the vector, i.e.

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|\ |\mathbf{a}|\cos 0^\circ = |\mathbf{a}|^2.$$

The converse of the first property gives an interesting result:

If $\mathbf{a}\cdot\mathbf{b} = 0$, then either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or *the direction of* **a** *is perpendicular to the direction of* **b**.

You should note that this is slightly different from the multiplication of ordinary numbers, where $xy = 0$ implies either x or y is zero. For the dot product of two vectors the expression $\mathbf{a}\cdot\mathbf{b} = 0$ gives an extra possibility, i.e. the angle between **a** and **b** is 90°.

Example 1

a and **b** are two vectors given in Cartesian component form by $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = -4\mathbf{i} + 2\mathbf{j}$ and $\mathbf{a}\cdot\mathbf{b} = 0$. What can we deduce about **a** and **b**?

Solution

Since neither **a** nor **b** is zero, we know that **a** is perpendicular to **b**. Figure 3 shows arrows representing **a** and **b**, and the angle between them is 90°.

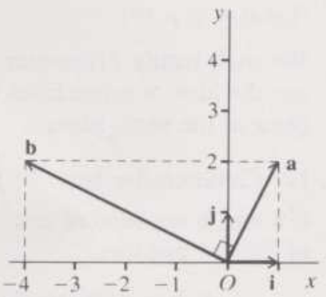


Figure 3

We shall now interpret the dot product geometrically by considering the projection of a vector.

3.2 The projection of a vector

A vector **b** is represented in Figure 4 by the arrow $\overrightarrow{OB}$ and makes an angle θ with another vector **a** represented by $\overrightarrow{OA}$. The length OP is defined to be the **projection of the vector b onto the line** OA, and this projection is a scalar quantity. Its value is $|\mathbf{b}|\cos\theta$. *Note that it will be negative if* $\theta > 90^\circ$, *i.e. if P and A lie on opposite sides of O.*

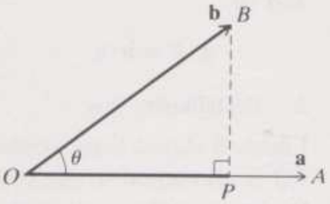


Figure 4

Example 2

The Cartesian components of a vector are its projections onto the x- and y-axes. If $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then:

b_1 is the projection of **b** onto Ox (i.e. length OP in Figure 5);

b_2 is the projection of **b** onto Oy (i.e. length OQ in Figure 5).

Referring again to Figure 4, the projection of **b** onto the line OA is OP and $OP = |\mathbf{b}|\cos\theta$; thus we may interpret the dot product $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta$ as the magnitude of **a** multiplied by the projection of **b** onto the direction of **a**.

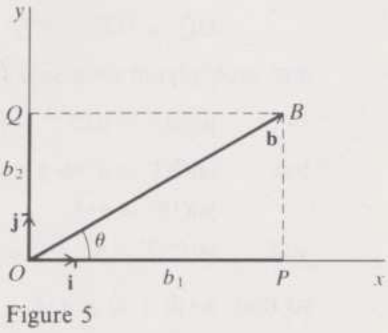


Figure 5

For instance, in the example above we could write

$$b_1 = \mathbf{i} \cdot \mathbf{b} \quad \text{and} \quad b_2 = \mathbf{j} \cdot \mathbf{b},$$

as the magnitudes of **i** and **j** are both equal to 1.

We can equally well use the facts that $OP = |\mathbf{b}| \cos\theta$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ to express the projection in terms of the dot product:

If $\overrightarrow{OA} = \mathbf{a}$, then the projection of **b** onto the line OA is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

3.3 Properties of the dot product

In Section 1 we established the algebraic laws for the scaling and addition of vectors; we now formulate the algebraic laws for the dot product of vectors.

Properties of the dot product

0. $\mathbf{a} \cdot \mathbf{b}$ is a scalar.

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, i.e. the dot product is commutative.

2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, i.e. the dot product is distributive over addition.

3. $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$.

4. If $\mathbf{a} \cdot \mathbf{b} = 0$ and **a** and **b** are not zero vectors, then **a** is perpendicular to **b**.

Exercise 2

Use Properties 1 and 2 to show that $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.

[*Solution on p. 50*]

We shall justify Properties 1, 3 and 4 from the definition of the dot product and use the idea of projections to illustrate Property 2 for **coplanar** vectors (i.e. vectors lying in the same plane).

1. Commutative law

If **a** and **b** are vectors and θ is the angle between **a** and **b**, then from the definition of the dot product,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta \quad \text{and} \quad \mathbf{b} \cdot \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}| \cos\theta,$$

and so

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

2. Distributive law

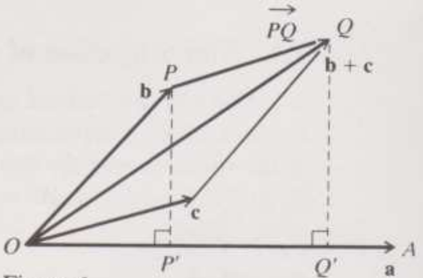


Figure 6

Figure 6 shows three coplanar vectors **a**, **b** and **c** and the sum $\mathbf{b} + \mathbf{c}$. The vector $\overrightarrow{PQ}$ is also equal to the vector **c**, so that the projection of $\overrightarrow{PQ}$ onto OA (i.e. $P'Q'$) is the same as the projection of **c** onto OA. The figure also shows the projections of **b** onto OA (i.e. OP') and $\mathbf{b} + \mathbf{c}$ onto OA (i.e. OQ'). It is clear from the figure that

$$OQ' = OP' + P'Q'$$

and multiplying each side by $|\mathbf{a}|$ we get

$$|\mathbf{a}|OQ' = |\mathbf{a}|OP' + |\mathbf{a}|P'Q'.$$

But $|\mathbf{a}|OQ' = \mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$,

$|\mathbf{a}|OP' = \mathbf{a} \cdot \mathbf{b}$,

and $|\mathbf{a}|P'Q' = \mathbf{a} \cdot \overrightarrow{PQ} = \mathbf{a} \cdot \mathbf{c}$,

so that $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

We have established that the dot product is distributive over vector addition for the case of three *coplanar* vectors. The result is however true for *any* three vectors.

3. Scaling law

To establish the third property we consider the two cases $m > 0$ and $m < 0$ separately.

(*a*) $m > 0$

$m\mathbf{a}$ is a vector of magnitude $m|\mathbf{a}|$ and has the same direction as $\mathbf{a}$: if the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then the angle between $m\mathbf{a}$ and $\mathbf{b}$ is also θ.

Thus

$$(m\mathbf{a})\cdot\mathbf{b} = |m\mathbf{a}|\ |\mathbf{b}|\cos\theta = m|\mathbf{a}|\ |\mathbf{b}|\cos\theta,$$

i.e. $\quad (m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}).$

(*b*) $m < 0$

$m\mathbf{a}$ is a vector of magnitude $|m|\ |\mathbf{a}|$ but in the opposite direction to $\mathbf{a}$, so the angle between $m\mathbf{a}$ and $\mathbf{b}$ is $\pi - \theta$. Thus

$$\begin{aligned}(m\mathbf{a})\cdot\mathbf{b} &= |m\mathbf{a}|\ |\mathbf{b}|\cos(\pi-\theta) = |m|\ |\mathbf{a}|\ |\mathbf{b}|\cos(\pi-\theta)\\ &= -m|\mathbf{a}|\ |\mathbf{b}|\ (-\cos\theta)\\ &= m|\mathbf{a}|\ |\mathbf{b}|\cos\theta,\end{aligned}$$

i.e. $\quad (m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}).$

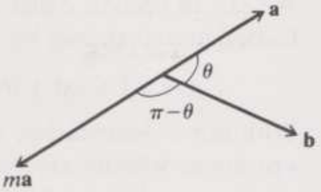


Figure 7

We have already met Property 4 in Subsection 3.1. The other three properties of the dot product correspond with some of the laws of algebra for ordinary numbers. They allow us to manipulate vector expressions containing additions, scalings and dot products of vectors.

Example 3

Expand the expression $\mathbf{x}\cdot\mathbf{y}$, given that $\mathbf{x} = 2\mathbf{u} + \mathbf{v}$ and $\mathbf{y} = \mathbf{u} - 5\mathbf{v}$. Calculate its value, given that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors.

Solution

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= (2\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-5\mathbf{v})\\ &= (2\mathbf{u})\cdot(\mathbf{u}-5\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}-5\mathbf{v}) && \text{(Exercise 2)}\\ &= (2\mathbf{u})\cdot\mathbf{u} - (2\mathbf{u})\cdot(5\mathbf{v}) + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot(5\mathbf{v}) && \text{(Property 2)}\\ &= 2(\mathbf{u}\cdot\mathbf{u}) - 10(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{u}) - 5(\mathbf{v}\cdot\mathbf{v}) && \text{(Property 3)}\\ &= 2(\mathbf{u}\cdot\mathbf{u}) - 9(\mathbf{u}\cdot\mathbf{v}) - 5(\mathbf{v}\cdot\mathbf{v}) && \text{(Property 1).}\end{aligned}$$

Now $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|\ |\mathbf{u}| = 1$ and $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|\ |\mathbf{v}| = 1$, since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Furthermore, $\mathbf{u}\cdot\mathbf{v} = 0$, since $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors (i.e. the angle between them is 90°). So we have

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= 2 - 0 - 5\\ &= -3.\end{aligned}$$

Exercise 3

(i) Expand the expression $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$.

(ii) Expand the expression $|\mathbf{a}+\mathbf{b}|^2$.

(iii) Given that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular unit vectors, find the value of m such that the two vectors $2\mathbf{a} + 3\mathbf{b}$ and $m\mathbf{a} + \mathbf{b}$ are perpendicular.

(iv) If $\mathbf{c} = 3\mathbf{u} + 5\mathbf{v}$, and $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors, what is the value of $|\mathbf{c}|$?

[*Solution on p. 50*]

The result of Exercise 3(iv) has an important generalization. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors, then we can calculate $|m\mathbf{u} + n\mathbf{v}|$ for any scalars m and n, as follows:

$$\begin{aligned}|m\mathbf{u}+n\mathbf{v}|^2 &= (m\mathbf{u}+n\mathbf{v})\cdot(m\mathbf{u}+n\mathbf{v})\\ &= m^2(\mathbf{u}\cdot\mathbf{u}) + 2mn(\mathbf{u}\cdot\mathbf{v}) + n^2(\mathbf{v}\cdot\mathbf{v}) \qquad \text{(expanding as before)}\\ &= m^2 + n^2 \qquad (\text{since } \mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} = 1 \text{ and } \mathbf{u}\cdot\mathbf{v} = 0).\end{aligned}$$

Thus we obtain the following formula.

If **u** and **v** are perpendicular unit vectors, then

$$|m\mathbf{u} + n\mathbf{v}| = \sqrt{m^2 + n^2}.$$

Finally, a word of caution; $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is not in general the same as $\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$. The vector $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is a scaling of **c** by the number $\mathbf{a} \cdot \mathbf{b}$, whereas $\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ is a scaling of **a** by the number $\mathbf{b} \cdot \mathbf{c}$. Clearly these two vectors are not generally even parallel, let alone equal. For example, if $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$, then

$$(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (\mathbf{i} \cdot \mathbf{i})\mathbf{j} = \mathbf{j}$$

but $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = \mathbf{i}(\mathbf{i} \cdot \mathbf{j}) = \mathbf{0}$.

$\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ means the same as $(\mathbf{b} \cdot \mathbf{c})\mathbf{a}$; in general if m is a scalar and **a** is a vector, we can write $m\mathbf{a}$ or $\mathbf{a}m$ as convenient.

3.4 The dot product of vectors given in component form

We saw in Section 2 that an arbitrary vector **a** may be expressed in terms of the Cartesian unit vectors by

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

With this representation, vector addition and scaling become simple algebraic operations without any reference to diagrams. The definition of the dot product was expressed in terms of the magnitudes of the two vectors and the angle between them. We shall now see how to express the dot product in terms of the components of the two vectors.

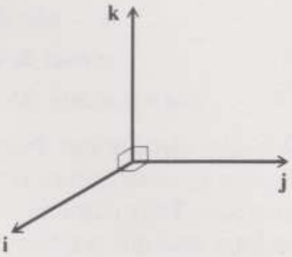


Figure 8

First, let us observe that **i**, **j** and **k** are unit vectors which are perpendicular to each other. Thus,

$$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{i} \cdot \mathbf{k} = 0 \quad \text{and} \quad \mathbf{j} \cdot \mathbf{k} = 0$$

and $\mathbf{i} \cdot \mathbf{i} = 1, \quad \mathbf{j} \cdot \mathbf{j} = 1 \quad \text{and} \quad \mathbf{k} \cdot \mathbf{k} = 1.$

Exercise 4

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, find the values of $\mathbf{a} \cdot \mathbf{i}$, $\mathbf{a} \cdot \mathbf{j}$ and $\mathbf{a} \cdot \mathbf{k}$.

[*Solution on p. 50*]

The solution to this exercise agrees with what we said in Subsection 3.2, that the components of a vector are projections of the vector onto the axes. If two vectors **a** and **b** have component forms $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then the dot product of **a** and **b** may be written as

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}),$$

and making use of the distributive rule (Property 2) and the result of Exercise 4, this reduces to

$$\boxed{\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.}$$

Exercise 5

If $\mathbf{a} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, show that $\mathbf{a} \cdot \mathbf{b} = -4$. What does the negative sign tell us?

[*Solution on p. 50*]

The component form of the dot product has an important application in calculating the angle between two vectors. You have already seen that if $\mathbf{a} \cdot \mathbf{b} = 0$ and neither **a** nor **b** is zero, then **a** and **b** are perpendicular. For instance, if $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j}$, then $\mathbf{a} \cdot \mathbf{b} = (2 \times 2) + (-1 \times 4) = 0$ and so the angle between **a** and **b** is 90°. In general we do not have $\mathbf{a} \cdot \mathbf{b} = 0$, but using the equation defining the dot product of **a** and **b**, i.e. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, we have the following simple expression for finding the angle between **a** and **b**, assuming that **a** and **b** are non-zero vectors:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \text{ provided that neither } \mathbf{a} \text{ nor } \mathbf{b} \text{ is zero.}$$

Example 4

(i) Find the angle between the vector $\mathbf{a} = \mathbf{i} + \sqrt{3}\ \mathbf{k}$ and the x-axis.

(ii) Find the angle between the two vectors $\mathbf{a} = \mathbf{i} + \sqrt{3}\ \mathbf{k}$ and $\mathbf{b} = \sqrt{3}\ \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, and show that $\mathbf{c} = -2\sqrt{3}\ \mathbf{i} + 2\mathbf{k}$ is perpendicular to $\mathbf{a}$.

Solution

(i) The direction of the x-axis is $\mathbf{i}$, and so the angle between $\mathbf{a} = \mathbf{i} + \sqrt{3}\ \mathbf{k}$ and $\mathbf{i}$ is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}|} = a_1/|\mathbf{a}|,$$

where a_1 is the x-component of $\mathbf{a}$ (i.e. $a_1 = 1$).

But $|\mathbf{a}| = \sqrt{1+3} = 2;$

thus $\cos\theta = \frac{1}{2}$ and θ, the angle between $\mathbf{a}$ and the x-axis, is 60°.

(ii) We have

$$\mathbf{a} = \mathbf{i} + \sqrt{3}\ \mathbf{k} \quad\text{and}\quad \mathbf{b} = \sqrt{3}\ \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}.$$

Thus $|\mathbf{a}| = \sqrt{1+3} = 2, \qquad |\mathbf{b}| = \sqrt{3+4+9} = 4$

and $\mathbf{a}\cdot\mathbf{b} = (1 \times \sqrt{3}) + (0 \times -2) + (\sqrt{3} \times 3) = 4\sqrt{3}.$

So $\cos\theta = \dfrac{4\sqrt{3}}{2 \times 4} = \dfrac{\sqrt{3}}{2},$

and therefore

$$\theta = 30°.$$

To test whether $\mathbf{a}$ and $\mathbf{c}$ are perpendicular, we calculate the dot product of $\mathbf{a}$ and $\mathbf{c}$.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{c} &= (\mathbf{i} + \sqrt{3}\ \mathbf{k})\cdot(-2\sqrt{3}\ \mathbf{i} + 2\mathbf{k})\\ &= (1 \times -2\sqrt{3}) + (0 \times 0) + (\sqrt{3} \times 2)\\ &= 0.\end{aligned}$$

Since $\mathbf{a}\cdot\mathbf{c} = 0$, and $\mathbf{a}$ and $\mathbf{c}$ are non-zero vectors, then $\mathbf{c}$ is perpendicular to $\mathbf{a}$.

Exercise 6

(i) Vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \quad\text{and}\quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

Find the magnitudes of $\mathbf{a}$ and $\mathbf{b}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$.

(ii) Which of the following vectors is perpendicular to the vector $\mathbf{a}$ in (i)?

$$\mathbf{c} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{d} = -2\mathbf{i} + \mathbf{k}, \quad \mathbf{e} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

(iii) Find the projection of the vector $\mathbf{a} + 2\mathbf{b}$ ($\mathbf{a}$ and $\mathbf{b}$ as in (i)) onto the line joining the origin to the point (1, 1, 1).

The definition of the projection of a vector is on p. 21.

[*Solution on p. 50*]

3.5 The cross product

This is another product of two vectors, and is defined to be *a vector*, whose direction is perpendicular to the two given vectors. This is in contrast to the dot product, which is a scalar.

Suppose that we are given two vectors $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle between their directions; then the **cross product** of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined to be the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\ |\mathbf{b}| \sin\theta)\mathbf{c}$$

with magnitude $|\mathbf{a}|\ |\mathbf{b}| \sin\theta$ and direction defined by $\mathbf{c}$, where $\mathbf{c}$ is a unit vector whose direction is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. The sense of $\mathbf{c}$ is given by the

right hand screw rule as shown in Figure 9, where **c** is perpendicular to the plane containing **a** and **b**. (The television programme shows an alternative rule for remembering the direction of **c**. It uses the thumb and first two fingers of the right hand, as in Subsection 2.2, and is called the **right hand rule**. If the vector **a** points along the thumb and the vector **b** points along the first finger, then the vector **c** points along the middle finger. Both rules lead to the same sense for the vector **c**, and you should use the rule with which you feel most comfortable.)

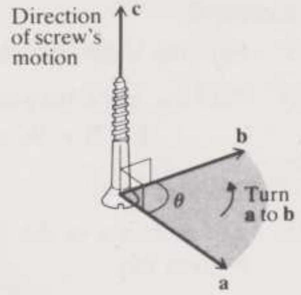


Figure 9

The cross product $\mathbf{a} \times \mathbf{b}$ (pronounced '**a** cross **b**') is also called the **vector product**, which stresses the fact that $\mathbf{a} \times \mathbf{b}$ is a vector. The order of writing down **a** and **b** is very important. According to the screw rule, $\mathbf{b} \times \mathbf{a}$ is a vector in the opposite direction to $\mathbf{a} \times \mathbf{b}$. Figure 10 shows what would happen to the screw if we turned from **b** to **a**. The unit vector **d** in the direction of $\mathbf{b} \times \mathbf{a}$ is in the opposite sense to **c**, i.e.

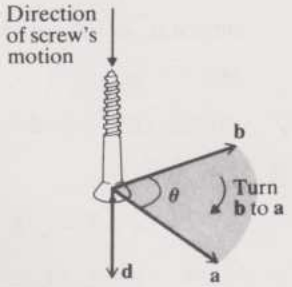


Figure 10

$$\mathbf{d} = -\mathbf{c}.$$

Hence

$$\mathbf{b} \times \mathbf{a} = (|\mathbf{b}|\,|\mathbf{a}| \sin\theta)\mathbf{d} = -(|\mathbf{b}|\,|\mathbf{a}| \sin\theta)\mathbf{c},$$

and therefore

$$\boxed{\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}).}$$

Exercise 7

Three vectors **u**, **v** and **w** of magnitudes 2, 3 and 4 units respectivley lie in the xy-plane, and the directions of the vectors make angles 30°, 60° and 30° respectively to the positive x-axis, turning towards the positive y-axis. Use the definition of cross product to find the vectors **u** $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \times \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$.

[*Solution on p. 51*]

This exercise illustrates an important property of the cross product. If two vectors **a** and **b** are parallel, then the angle between their directions is zero, and so the cross product of **a** and **b** is the zero vector, because the magnitude of the vector, i.e. $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$, is zero. This property gives us a method of testing for parallel vectors:

> If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, and **a** and **b** are not zero vectors, then **a** is parallel to **b**.

So, we can test for perpendicular vectors by using the dot product and for parallel vectors by using the cross product. In particular,

$$\boxed{\mathbf{a} \times \mathbf{a} = \mathbf{0} \text{ for any vector } \mathbf{a}.}$$

The following example shows a geometrical application of the cross product which will be useful in demonstrating the distributive property for cross products.

Example 5

Two vectors define a parallelogram as shown in Figure 11. Express the area of the parallelogram in terms of $\mathbf{a} \times \mathbf{b}$.

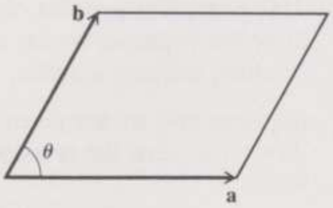


Figure 11

Solution

The area of the parallelogram is given by $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$, and this is the magnitude of $\mathbf{a} \times \mathbf{b}$. The direction of $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane of the parallelogram and with sense 'out of' the paper. We can associate with the parallelogram a vector 'area' **A**, where

$$\mathbf{A} = \mathbf{a} \times \mathbf{b} \quad \text{and} \quad |\mathbf{A}| = |\mathbf{a} \times \mathbf{b}|.$$

3.6 Properties of the cross product

The following are two of the most important algebraic properties of the cross product.

1. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ and
   $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$,

   i.e. the cross product is distributive over vector addition.
2. $(m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (m\mathbf{b})$.

Property 2 is quite easy:

2. Scaling law

If $\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\,|\mathbf{b}| \sin\theta)\mathbf{c}$, then $m(\mathbf{a} \times \mathbf{b})$ is just a scaling of the vector $\mathbf{c}$, i.e.

$$m(\mathbf{a} \times \mathbf{b}) = (|\mathbf{a}|\;|\mathbf{b}| \sin\theta)m\mathbf{c};$$

and $(m\mathbf{a}) \times \mathbf{b}$ is the cross product of a scaling of $\mathbf{a}$ and $\mathbf{b}$, i.e.

$$(m\mathbf{a}) \times \mathbf{b} = (m|\mathbf{a}|\;|\mathbf{b}| \sin\theta)\mathbf{c} = (|\mathbf{a}|\;|\mathbf{b}| \sin\theta)m\mathbf{c}.$$

Similarly,

$$\mathbf{a} \times (m\mathbf{b}) = (|\mathbf{a}|\; m|\mathbf{b}| \sin\theta)\mathbf{c} = (|\mathbf{a}|\,|\mathbf{b}| \sin\theta)m\mathbf{c}.$$

Finally, we shall illustrate Property 1 for coplanar vectors, by using the geometric interpretation of cross products which we found in Example 5.

1. Distributive law

Figure 12 shows the three parallelograms $OSTR$, $SPQT$ and $OPQR$, in the same plane, whose areas are equal to $|\mathbf{a} \times \mathbf{b}|$, $|\mathbf{a} \times \mathbf{c}|$ and $|\mathbf{a} \times (\mathbf{b} + \mathbf{c})|$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors lying in the same plane. The triangles OSP and RTQ are made up of the same vectors ($\overrightarrow{RT} = \mathbf{b}$, $\overrightarrow{TQ} = \mathbf{c}$ and $\overrightarrow{RQ} = \mathbf{b} + \mathbf{c}$), and so have the same area.

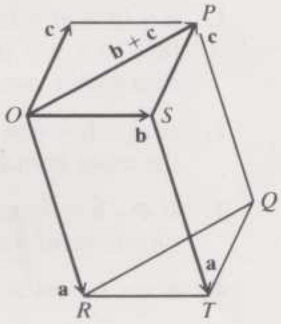


Figure 12

Thus the area of $OPQR$ is equal to the area of $OSPQTR$, or in vector form

$$|\mathbf{a} \times (\mathbf{b} + \mathbf{c})| = |\mathbf{a} \times \mathbf{b}| + |\mathbf{a} \times \mathbf{c}|.$$

The direction of each of the three vectors $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ is the same (by the definition of the cross product, each cross product has direction perpendicular to the plane of the figure pointing 'out of the page'), and therefore in this special case of coplanar vectors,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}).$$

Similarly, we could show (again for this special case) that

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

Although we have demonstrated this distributive rule for coplanar vectors, it is true for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We can now use these rules to manipulate algebraic expressions involving cross products.

Example 6

Expand and simplify $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$.

Solution

We have, using the distributive law,

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) &= (\mathbf{a} \times (\mathbf{a} - \mathbf{b})) + (\mathbf{b} \times (\mathbf{a} - \mathbf{b}))\\ &= (\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times (-\mathbf{b})) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times (-\mathbf{b}))\\ &= (\mathbf{a} \times \mathbf{a}) - (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{b}) && \text{scaling law}\\ &= -(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) && \mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = \mathbf{0}\\ &= -2(\mathbf{a} \times \mathbf{b}). && \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}\end{aligned}$$

Exercise 8

Expand and simplify:

(i) $(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (\mathbf{b} - \mathbf{c})$

(ii) $\mathbf{d} \times \mathbf{e}$ where $\mathbf{d} = 2\mathbf{u} - \mathbf{v}$ and $\mathbf{e} = \mathbf{u} + 2\mathbf{v}$.

[*Solution on p. 51*]

Two of our usual laws of algebra, the commutative and associative rules, are missing from our list of properties for the cross product. The reason for their omission is that *they are not true*. That is to say,

$$\boxed{\begin{gathered}\mathbf{b} \times \mathbf{a} \neq \mathbf{a} \times \mathbf{b},\\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.\end{gathered}}$$

So we have to be very careful when manipulating expressions containing cross products. In particular, the expression $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ has no meaning; you must put in brackets.

Exercise 9

(i) What is $\mathbf{b} \times \mathbf{a}$ in terms of $\mathbf{a} \times \mathbf{b}$?

(ii) Demonstrate that the cross product is not associative by considering $\mathbf{a} \times (\mathbf{b} \times \mathbf{b})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

[*Solution on p. 51*]

In the following box we have collected together the properties of the cross product discussed in Subsections 3.5 and 3.6.

Properties of the cross product

0. $\mathbf{a} \times \mathbf{b}$ is a vector.

1. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$, i.e. the cross product is distributive over vector addition.

2. $(m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (m\mathbf{b})$, i.e. the scaling properties of the cross product are similar to those of the dot product.

3. If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ and $\mathbf{a}$ and $\mathbf{b}$ are not zero vectors, then the directions of $\mathbf{a}$ and $\mathbf{b}$ are parallel.

4. $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.

5. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

6. In general, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

3.7 The cross product of vectors given in component form

Exercise 10

(i) Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

(ii) What are the values of $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$ and $\mathbf{k} \times \mathbf{k}$?

(iii) Expand and simplify $(\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $(\mathbf{i} \times (\mathbf{i} + \mathbf{k})) - ((\mathbf{i} + \mathbf{j}) \times \mathbf{k})$.

[*Solution on p. 51*]

Exercise 10 illustrates the cross product of expressions involving **i**, **j** and **k**. The fact that $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ follows immediately from the parallel vector rule. The cyclic pattern of the products $\mathbf{i} \times \mathbf{j}$, $\mathbf{j} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$ is shown in Figure 13.

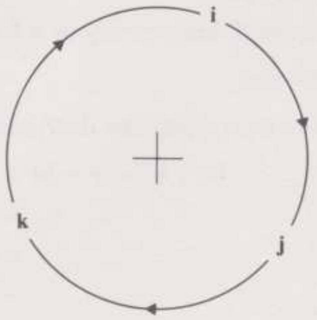


Figure 13

For example, if we go round the circle clockwise we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

If we go in an anticlockwise direction the cross products are negative. For example,

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}.$$

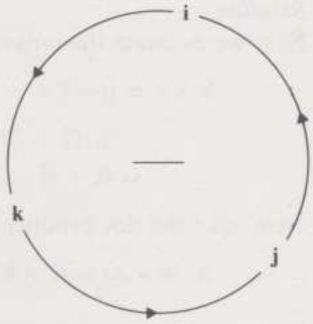


Figure 14

Given two vectors **a** and **b** in Cartesian component form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}),$$

which on expansion and simplification using the results of Exercise 10 reduces to

$$\boxed{\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.}$$

Example 7

If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$

Solution

Since $a_1 = 2$, $a_2 = 1$, $a_3 = -1$ and $b_1 = 1$, $b_2 = -3$, $b_3 = 4$, the formula above gives

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \big((1 \times 4) - (-1 \times -3)\big)\mathbf{i} + \big((-1 \times 1) - (2 \times 4)\big)\mathbf{j} \\ &\quad + \big((2 \times -3) - (1 \times 1)\big)\mathbf{k} \\ &= \mathbf{i} - 9\mathbf{j} - 7\mathbf{k}.\end{aligned}$$

Exercise 11

If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = -4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$, then find $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$. From your results what can you say about **a** and **c**?

Exercise 12

If $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, find a unit vector whose direction is perpendicular to the directions of both **a** and **b**.

[*Solutions to Exercises 11 and 12 on p. 51*]

3.8 Dots and crosses (Television and Tape subsection)

TV 14

The television programme uses combinations of dot and cross products and sets a problem which is taken up by the tape. So we start this subsection with some exercises to give you further practice at manipulating expressions containing dot and cross products. You should set aside sufficient time to work through the tape problem immediately after watching the television programme.

Exercise 13

If $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, then find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. What is the essential difference between these two products?

What is the value of the dot product of **a** with the vector $\mathbf{a} \times \mathbf{b}$ which you have just calculated?

[*Solution on p. 52*]

This last result, $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$, should not surprise you; $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the vector **a**, and so $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ is the dot product of two perpendicular vectors, which is zero. Now $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ is a combination of three vectors. It is a special case of a product called the **triple scalar product**, defined in the following way:

> If **a**, **b** and **c** are three vectors, then the **triple scalar product** of **a**, **b** and **c** is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

We evaluate $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ by first working out the cross product $\mathbf{b} \times \mathbf{c}$ and then the dot product of **a** with $(\mathbf{b} \times \mathbf{c})$. The following example illustrates this.

Example 8

If $\mathbf{a} = \mathbf{i} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{c} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, what is the value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$?

Solution

First we evaluate the cross product $\mathbf{b} \times \mathbf{c}$. We have

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= ((-1 \times -4) - (0 \times 3))\mathbf{i} + ((0 \times 1) - (2 \times -4))\mathbf{j} \\ &\quad + ((2 \times 3) - (-1 \times 1))\mathbf{k} \\ &= 4\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}.\end{aligned}$$

Now take the dot product of $\mathbf{a}$ with this vector:

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{i} + \mathbf{k}) \cdot (4\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}) &= 4 + 0 + 7 \\ &= 11.\end{aligned}$$

Exercise 14

For the three vectors $\mathbf{a} = -\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{k}$, find the value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

[*Solution on p. 52*]

The television programme for this unit takes a close look at the dot and cross products of two vectors, and uses these products to find the shortest distance between two skew lines.

The programme can be summarized as follows:

Part 1: a review of the idea of projections onto lines and planes and how projections can be written down in terms of the dot product.

Part 2: a review of the definition of the cross product of two vectors and the geometrical application of finding a vector which is perpendicular to a plane drawn through three points.

Part 3: the use of the dot and cross products to find the shortest distance between two skew lines.

In Part 1 of the programme a hexagon is used to illustrate the idea of projections. A problem based on the hexagon is left for you to solve after the programme with the help of the tape. There is no need to take a note of the details of the problem during the programme, because we will review them at the start of the tape.

Exercises 16 and 17 follow up the ideas in Part 2 of the programme.

The skew line problem is introduced at the start of the programme and considered in detail in Part 3. Two straight lines are said to be **skew lines** if they have the following two properties:

(i) the two lines are not parallel;

(ii) the two lines do not touch each other.

For example the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ in Figure 15 are skew, as we shall see in the television programme.

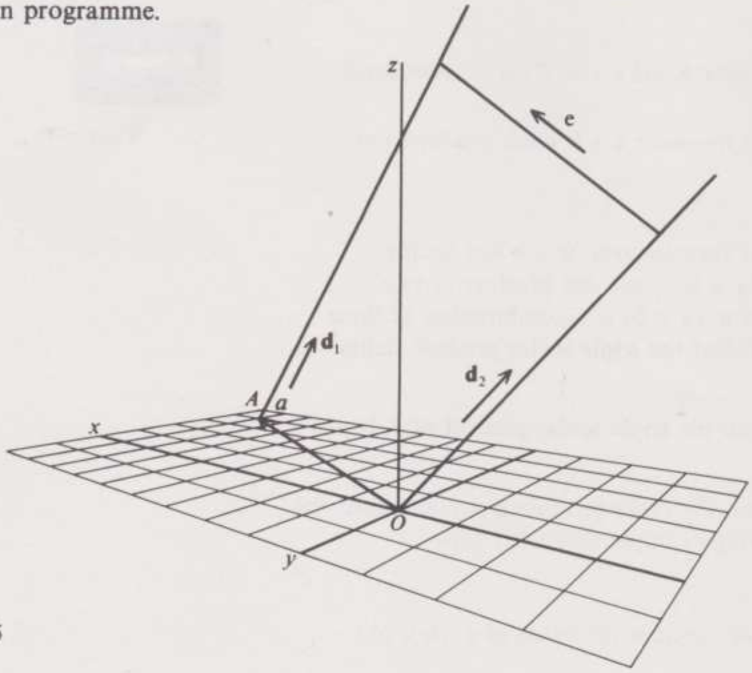


Figure 15

As the two skew lines do not touch, a problem of interest is to find the shortest distance between the two lines. Part 3 of the programme does this elegantly using vector methods. During the programme the results of the following exercise are used; so work through the exercise before watching.

Exercise 15

If $\mathbf{d}_1 = -5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{d}_2 = -3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{a} = 8\mathbf{i} - 4\mathbf{j}$, then evaluate $\dfrac{\mathbf{d}_1 \times \mathbf{d}_2}{|\mathbf{d}_1 \times \mathbf{d}_2|}$ and $\mathbf{a} \cdot \dfrac{\mathbf{d}_1 \times \mathbf{d}_2}{|\mathbf{d}_1 \times \mathbf{d}_2|}$.

[*Solution on p. 52*]

Now watch the television programme.

In the television programme, we introduced you to the problem of finding the angle between the sides of the hexagon. You are now going to solve that problem with the help of the tape.

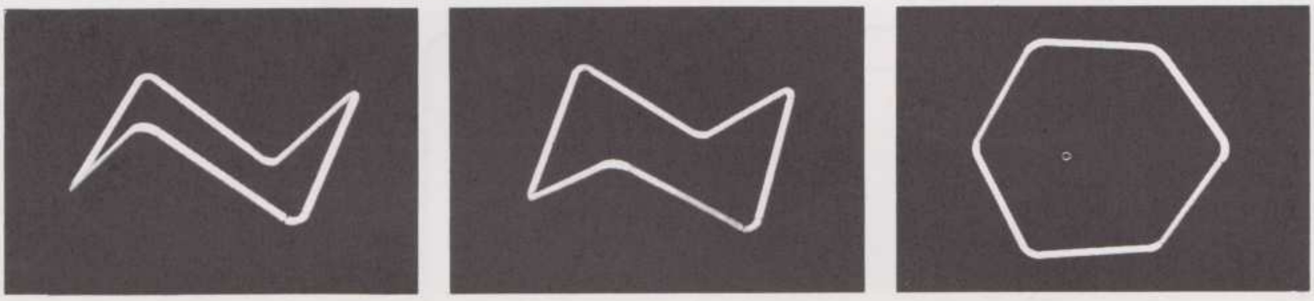
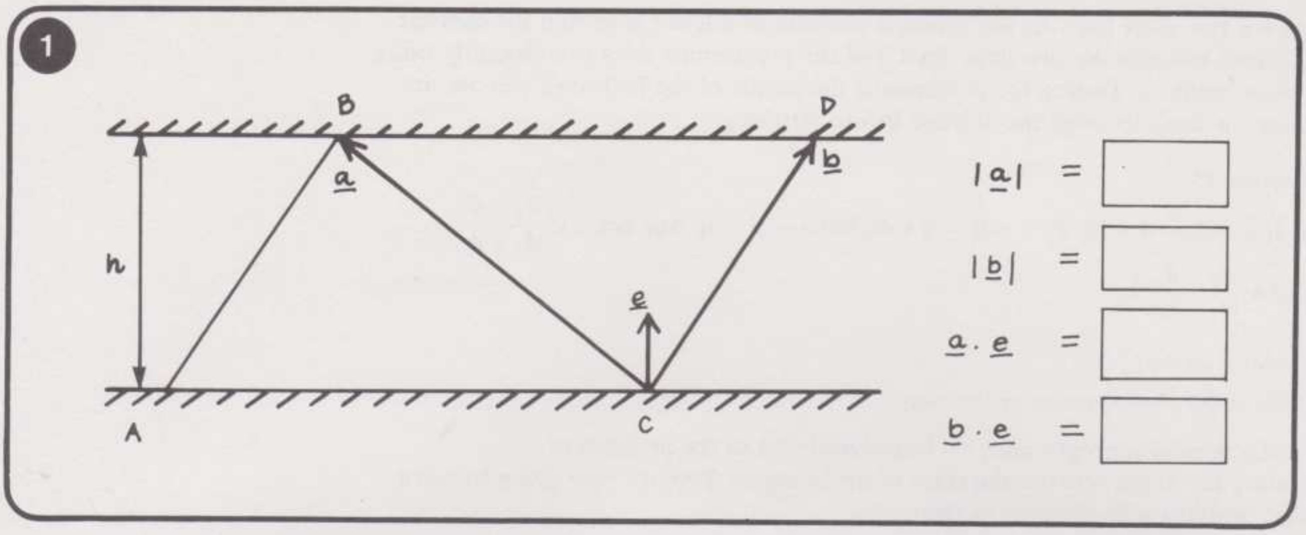
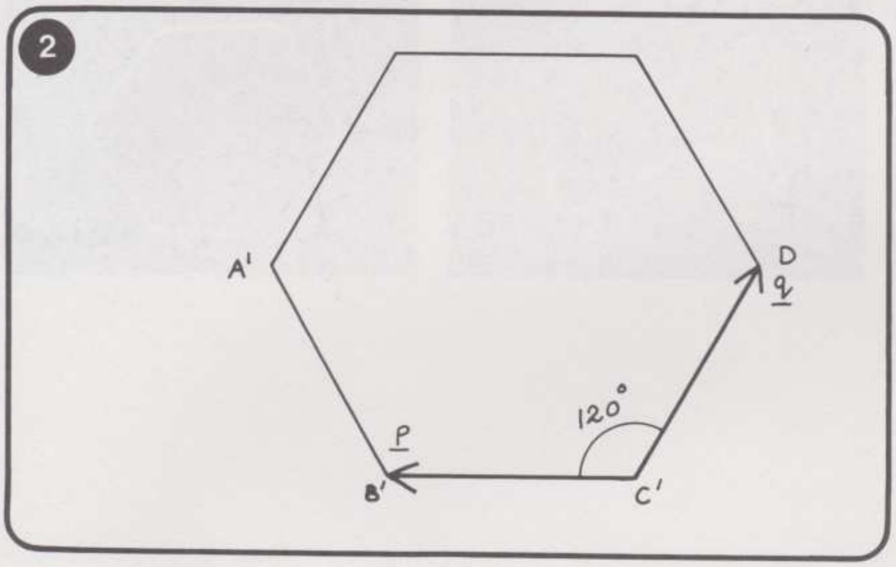
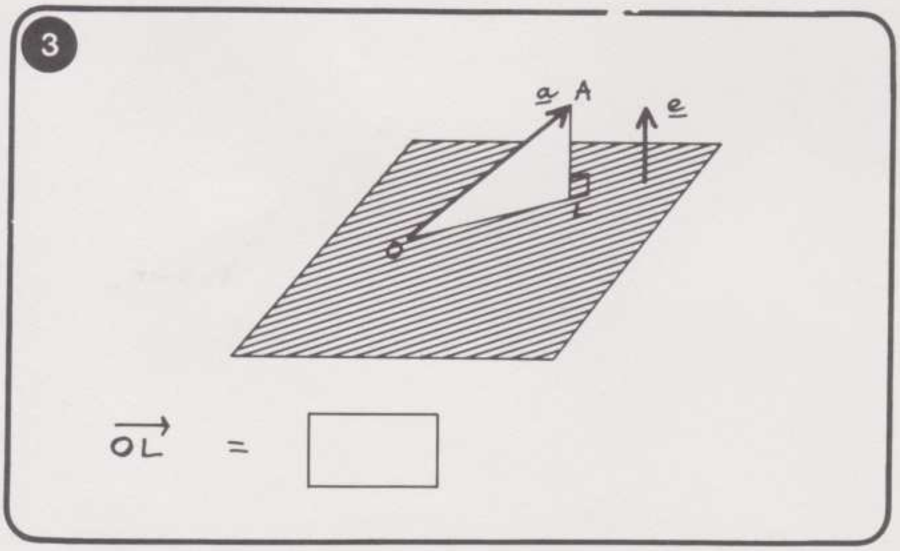


Figure 16

Now start the tape.


1
B
D
a
b
h
e
A
C
|a| =
|b| =
a · e =
b · e =


2
A'
D
q
120°
p
B'
C'


3
a
A
e
O
L
$\overrightarrow{OL}$ =

4

$$\underline{p} = \underline{a} - (\underline{a} \cdot \underline{e})\,\underline{e}$$

$$\underline{q} = \underline{b} - (\underline{b} \cdot \underline{e})\,\underline{e}$$

5

$\cos\theta =$

6

$$\underline{a} \cdot \underline{b} = (\underline{p} + (\underline{a} \cdot \underline{e})\,\underline{e}) \cdot (\underline{q} + (\underline{b} \cdot \underline{e})\,\underline{e})$$

$\underline{a} \cdot \underline{b} =$

7

$|\underline{p}| =$

$|\underline{q}| =$

8

$\underline{p} \cdot \underline{p} =$

$\underline{q} \cdot \underline{q} =$

9

$|\underline{p}| =$

$|\underline{q}| =$

$\underline{a} \cdot \underline{b} =$

$\cos\theta =$

Solutions to Frames 5, 6 and 8

5a

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

6a

$$\underline{a}\cdot\underline{b} = \underline{p}\cdot\underline{q} + \underline{p}\cdot((\underline{b}\cdot\underline{e})\,\underline{e})$$

$$+ ((\underline{a}\cdot\underline{e})\,\underline{e})\cdot\underline{q} + ((\underline{a}\cdot\underline{e})\,\underline{e})\cdot((\underline{b}\cdot\underline{e})\,\underline{e})$$

$$\underline{a}\cdot\underline{b} = -\tfrac{1}{2}|\underline{p}||\underline{q}| + (\underline{a}\cdot\underline{e})(\underline{b}\cdot\underline{e})$$

8a

$$\underline{p}\cdot\underline{p} = (\underline{a} - (\underline{a}\cdot\underline{e})\,\underline{e})\cdot(\underline{a} - (\underline{a}\cdot\underline{e})\,\underline{e})$$

$$\underline{p}\cdot\underline{p} = \underline{a}\cdot\underline{a} - \underline{a}\cdot((\underline{a}\cdot\underline{e})\,\underline{e})$$

$$- ((\underline{a}\cdot\underline{e})\,\underline{e})\cdot\underline{a} + ((\underline{a}\cdot\underline{e})\,\underline{e})\cdot((\underline{a}\cdot\underline{e})\,\underline{e})$$

$$\underline{p}\cdot\underline{p} = |\underline{a}|^2 - (\underline{a}\cdot\underline{e})^2$$

Finally we illustrate another idea from the television programme in the following exercise.

Exercise 16

A plane is drawn through the three points $A\ (2,1,0)$, $B\ (0,1,3)$ and $C\ (1,0,1)$.

(i) Using the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, find two vectors lying in the plane.

(ii) Hence find a vector perpendicular to this plane. (Such a vector is called a **normal vector**, and is denoted by $\mathbf{n}$.)

[Solution on p. 52]

We can go one step further with this last exercise and find an equation for the plane through the points A, B and C.

Suppose that P is any point in the plane with position vector

$$\overrightarrow{OP} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

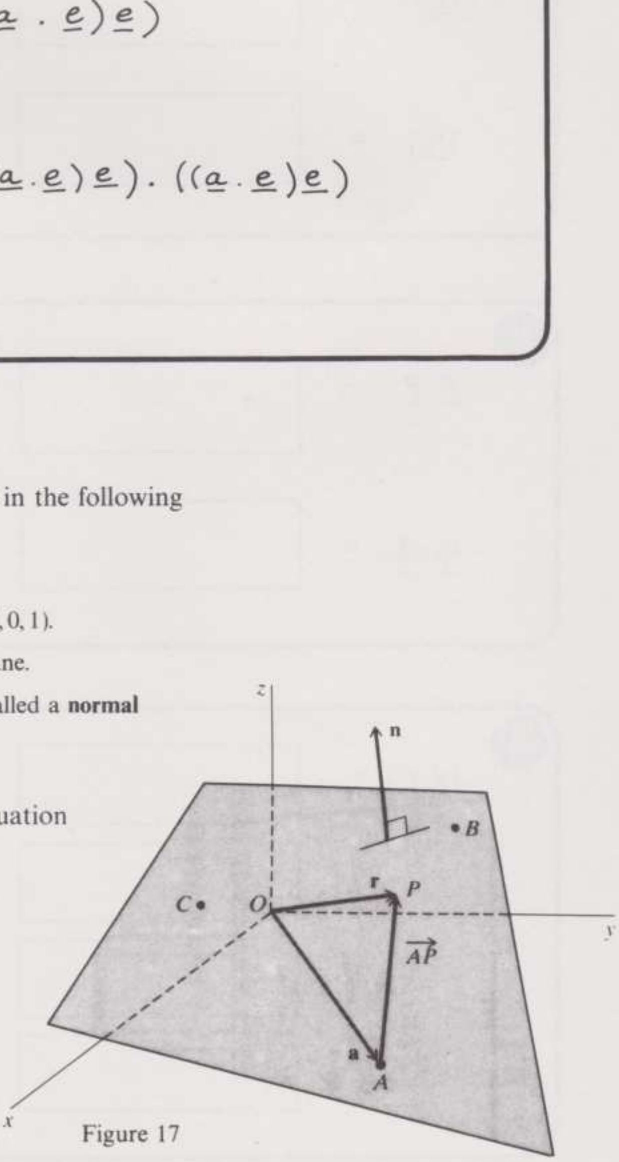


Figure 17

Then the vector $\mathbf{r} - \mathbf{a} = \overrightarrow{AP}$ lies in the plane and so is perpendicular to the normal vector **n**. Therefore

$$(\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = 0.$$

Now

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} \quad \text{and} \quad \mathbf{n} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k},$$

and so we have

$$(\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = \big((x-2)\mathbf{i} + (y-1)\mathbf{j} + z\mathbf{k}\big)\cdot(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 0$$

which reduces to

$$3(x-2) - (y-1) + 2z = 0$$

or

$$3x - y + 2z = 5.$$

This is the equation of the plane through the points A, B and C. More generally, the equation

$$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$$

is the equation of a plane with normal vector given by

$$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k},$$

and passing through the point with coordinates (a, b, c).

Exercise 17

The equation of a plane is given by

$$x - 2y + 5z = 7.$$

Find a unit vector perpendicular to this plane.

[*Solution on p. 52*]

Summary of Section 3

1. The **dot product** or **scalar product** of two vectors **a** and **b** is defined by

   $$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

   where θ is the angle between **a** and **b**. Two immediate consequences of this definition are:

   (i) $\mathbf{a}\cdot\mathbf{b}$ is a scalar;

   (ii) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$.

2. The algebraic properties of the dot product are given in the box on page 22.

3. If θ is the angle between **a** and **b**, then the expression $|\mathbf{b}|\cos\theta$ is called the **projection** of **b** onto the direction of the vector **a**, and $|\mathbf{a}|\cos\theta$ is the **projection** of **a** onto the direction of the vector **b**. Hence

   $$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}| \text{ times the projection of } \mathbf{b} \text{ onto the direction of } \mathbf{a}\\ &= |\mathbf{b}| \text{ times the projection of } \mathbf{a} \text{ onto the direction of } \mathbf{b}.\end{aligned}$$

4. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then:

   (i) $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$;

   (ii) $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2$;

   (iii) if θ is the angle between **a** and **b**, then

   $$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

5. The **cross product** or **vector product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\,|\mathbf{b}| \sin\theta)\mathbf{c}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{c}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$. The direction of $\mathbf{c}$ is found using the screw rule turning from $\mathbf{a}$ to $\mathbf{b}$.

6. The algebraic properties of the cross product are given in the box on page 28.

7. The Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have the following cross products among themselves:

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0} \quad \text{and} \quad \mathbf{k} \times \mathbf{k} = \mathbf{0},$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \text{and} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

8. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

9. If $\mathbf{a}$ and $\mathbf{b}$ are vectors lying in a plane, then $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane and is called a **normal vector**.

The equation

$$(x - a)n_1 + (y - b)n_2 + (z - c)n_3 = 0$$

is the equation of a plane whose normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$$

and which passes through the point (a, b, c).

End of section exercises

Exercise 18

(i) Is $\mathbf{a} \cdot \mathbf{b}$ a vector?

(ii) Can $\mathbf{a} \cdot \mathbf{b}$ be negative?

(iii) What would be special about $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$?

(iv) If $\mathbf{a} \cdot \mathbf{b} = 0$, what can we say about $\mathbf{a}$ and $\mathbf{b}$?

(v) If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what can we say about $\mathbf{a}$ and $\mathbf{b}$?

Exercise 19

Suppose that the vectors $\mathbf{r}$ and $\mathbf{s}$ are directed towards North and North-East respectively, and define $\mathbf{r} \times \mathbf{s} = \mathbf{t}$.

(i) What is the direction of $\mathbf{t}$? In what direction is $\mathbf{s} \times \mathbf{r}$?

(ii) In what direction is $\mathbf{t} \times \mathbf{r}$?

(iii) If $|\mathbf{r}| = |\mathbf{s}| = 1$, what is $|\mathbf{t}|$?

(iv) Calculate the vector $\mathbf{t} \times (\mathbf{r} \times \mathbf{s})$.

(v) If $|\mathbf{r}| = |\mathbf{s}| = 1$, what is the value of $\mathbf{r} \cdot \mathbf{s}$?

(vi) If $|\mathbf{r}| = |\mathbf{s}| = 1$, what is the value of $\mathbf{s} \cdot (\mathbf{t} \times \mathbf{r})$?

Exercise 20

Find the value of $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}$ for any non-zero vectors $\mathbf{a}$ and $\mathbf{b}$.

[*Solutions to Exercises 18–20 on pp. 52–53*]

4 Differentiating vectors

4.1 Vector functions

One of the main uses of vectors in applied mathematics is to describe the motion of a particle in space. The method used is a generalization of the way we describe motion on a line, which you already know from *Unit 4*. For motion on a line, we make this line the x-axis and then represent the position of the particle by a real number (x) which is the x-coordinate of the particle's position; then we can describe a motion of the particle by making this x-coordinate a variable which depends on another variable t representing the time. For example, the rule giving x in terms of t might be

$$x = 5t^2.$$

In general, when x depends on t we can write

$$x = x(t),$$

where $x(t)$ stands for some formula involving t but not x.

To describe motion in space we use the same idea, but now the position of the particle is expressed by a vector rather than a real number. Denoting this position vector by $\mathbf{r}$, we can describe a motion of the particle by making $\mathbf{r}$ a variable which depends on the (scalar) variable t representing the time. For example, the rule giving $\mathbf{r}$ in terms of t might be

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}.$$

If $\mathbf{r}$ depends on t in some unspecified way we may write

$$\mathbf{r} = \mathbf{r}(t),$$

where $\mathbf{r}$ stands for some general vector-valued expression involving t but not $\mathbf{r}$.

Relationships of this kind can be usefully described using the mathematical concept of a **function**. A function consists of three parts — a set called the **domain**, a set called the **codomain**, and a rule which to each element of the domain associates a unique element of the codomain. Here we are concerned with functions for which the domain is the set of all real numbers, and the codomain is the set of all (three-dimensional) vectors. We shall call functions of this kind **vector functions** and denote them, usually, by the symbol $\mathbf{f}$.

This idea is introduced in Block I of *M101* and *MS283*. See also the Preparatory Booklet.

The image of the real number t under the vector function $\mathbf{f}$ is denoted by $\mathbf{f}(t)$. We often describe a vector function simply by giving the formula for its image — for example we could say 'the (vector) function $t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$' to mean the vector function $\mathbf{f}$ such that, for all real values of t,

$$\mathbf{f}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}.$$

Just as in the case of real functions, we will normally simplify notation slightly, in the following way. A vector (e.g. a position vector $\mathbf{r}$) is theoretically a different entity from the rule describing its variation with time, so we should write

$$\mathbf{r} = \mathbf{f}(t).$$

In practice, though, we will refer to $\mathbf{r}$ as 'a function of t' and often write $\mathbf{r}(t)$ for the value of $\mathbf{r}$ at time t.

Example 1

The position vector of a stone falling freely with zero initial velocity is given by

$$\mathbf{r} = x(t)\mathbf{i} = \tfrac{1}{2}gt^2\mathbf{i}$$

where the x-axis points vertically downwards. The direction of the vector $\mathbf{r}$ is constant but its magnitude, i.e. $\frac{1}{2}gt^2$, is a function of t, the time.

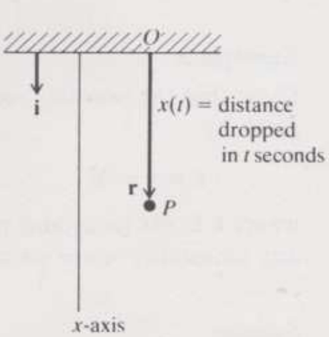


Figure 1

Example 2

In *Unit 7* you studied the motion of a simple pendulum by applying the conservation of energy equation. The *simple pendulum* consists of a particle P suspended from a fixed point O by an inextensible string. The body P moves along the arc of a circle with radius l, the length of the string. The position of P at any time t can be described by the Cartesian coordinates x and y, where $x = l\cos\theta$ and $y = l\sin\theta$; so that the position vector of P relative to O is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = (l\cos\theta)\mathbf{i} + (l\sin\theta)\mathbf{j}$$

where θ depends on t.

In this example the magnitude of $\mathbf{r}$ is constant (i.e. $|\mathbf{r}| = l$) and the direction of $\mathbf{r}$ (described by the angle θ) is a function of t.

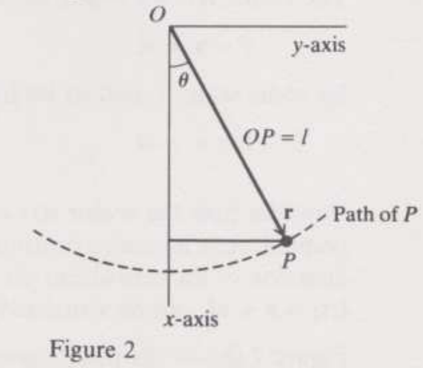


Figure 2

Exercise 1

Find the magnitudes and directions of the following vectors.

Which of the vectors have constant magnitude?

Which of the vectors have constant direction?

Is there a vector which does not depend on t?

(i) $\mathbf{a} = (1 + t^2)\mathbf{i}$

(ii) $\mathbf{b} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$

(iii) $\mathbf{c} = 4\mathbf{i} + 3\mathbf{j}$

(iv) $\mathbf{d} = e^t\mathbf{i} + 2e^t\mathbf{j}$.

[*Solution on p. 53*]

The vector $\mathbf{c}$ in the above exercise does not depend on t. This is apparent either from the components (which are just 4 and 3) or from the magnitude and direction of the vector (which are 5 and arc $\cos(\frac{4}{5})$ respectively).

> A vector for which *both* the magnitude *and* the direction are constant, or equivalently for which *each* component is constant, is called a **constant vector**.

The Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are constant vectors. However, not all unit vectors are constant vectors; for example, the vector $\mathbf{b}$ in Exercise 1 is a unit vector (its magnitude is 1), but it is *not* a constant vector because its direction depends on t.

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ do depend on t and so are functions of t.

4.2 Geometrical representation of a vector function

Suppose that $\mathbf{f}$ is a vector function, and consider a point P with position vector $\mathbf{r} = \mathbf{f}(t)$. (See Figure 3.) For different values of t the position of P changes because $\mathbf{f}(t)$ depends on t.

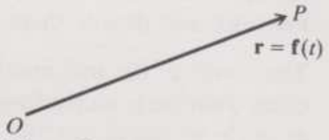


Figure 3

Figure 4 shows three positions of P for values of t equal to t_1, t_2 and t_3. Figure 5 shows that, as t varies continuously, the point P traces out a continuous curve in space.

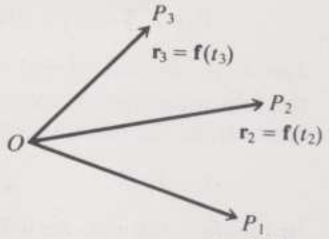

P_3
$\mathbf{r}_3 = \mathbf{f}(t_3)$
P_2
O
$\mathbf{r}_2 = \mathbf{f}(t_2)$
P_1
$\mathbf{r}_1 = \mathbf{f}(t_1)$

Figure 4

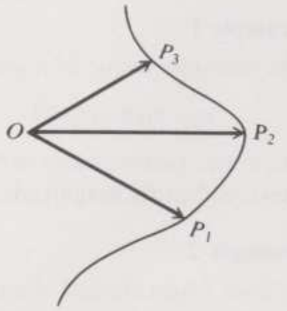


Figure 5

Example 3

Show that the position vector $\mathbf{r}$ of a general point P lying on a straight line is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{d}$$

where $\mathbf{a}$ is the (constant) position vector of a fixed point A lying on the line, $\mathbf{d}$ is any (constant) vector parallel to the line, and t is some scalar. (Figure 6)

Solution

The vector $\overrightarrow{AP} = \mathbf{r} - \mathbf{a}$ is parallel to $\mathbf{d}$, so we can write $\mathbf{r} - \mathbf{a}$ as a scaling of $\mathbf{d}$:

$$\mathbf{r} - \mathbf{a} = t\mathbf{d}$$

for some scalar t, and so we have

$$\mathbf{r} = \mathbf{a} + t\mathbf{d}.$$

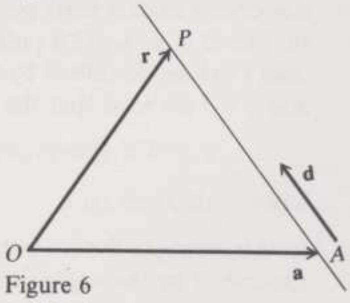


Figure 6

Consider now the vector $\mathbf{r}(t) = \mathbf{a} + t\mathbf{d}$. When $t = 0$, the point P coincides with the point A. As t increases continuously, the point P moves along the line in the direction of the arrowhead on the vector $\mathbf{d}$. So the vector function $\mathbf{f}$, where $\mathbf{f}(t) = \mathbf{a} + t\mathbf{d}$, can be visualized as a straight line in space.

Figure 7 shows the positions of the point P for values of t equal to -1, 0, 1 and 2.

The vectors $\overrightarrow{AP_1}$, $\overrightarrow{AP_3}$ and $\overrightarrow{AP_4}$ are just $-\mathbf{d}$, $\mathbf{d}$ and $2\mathbf{d}$ respectively.

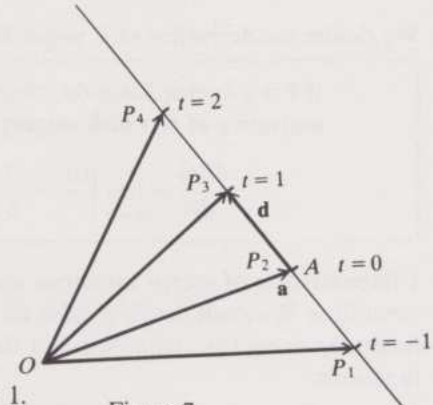


Figure 7

Example 4

Show that the image of the vector function

$$(\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \mathbf{k}$$

can be represented by a circle of radius 1 in the plane $z = 1$.

Solution

If we let the position vector $\mathbf{r}$ of a point P be given by

$$\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \mathbf{k},$$

then the z-component of $\mathbf{r}$ (i.e. the distance of P above the xy-plane) is always 1. Thus the path described by P, as t varies, always lies in the plane $z = 1$. We can find the path by considering the x- and y-components of $\mathbf{r}$ and eliminating t:

$$x = \cos t \quad \text{and} \quad y = \sin t$$

so that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

This is the equation of a circle of radius 1. The circle is drawn in the plane $z = 1$ and represents the path of P as t varies.

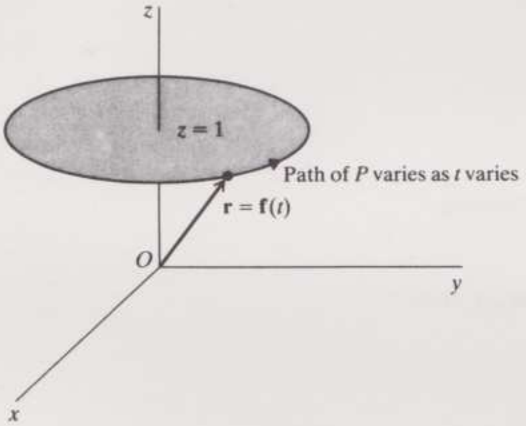

z
$z = 1$
Path of P varies as t varies
$\mathbf{r} = \mathbf{f}(t)$
O
y
x

Figure 8

To summarize, then, we can visualize a vector function $\mathbf{f}$ in terms of the curve that a point P, with position vector $\mathbf{f}(t)$, describes as t varies.

4.3 Differentiation of vector functions

You are familiar with the idea of a *scalar function* (that is, a function f for which $f(t)$ is a scalar—i.e. its codomain is the set of all real numbers). For instance, consider the motion of a particle moving in a straight line with constant acceleration a. The particle's position at any time may be denoted by $x(t)$ where

$$x(t) = x_0 + v_0 t + \tfrac{1}{2}at^2$$

and x_0 and v_0 are the initial position and velocity of the particle respectively. The velocity of the particle at any time, denoted by $v(t)$, is given in terms of the derivative of x with respect to t as

$$v(t) = \frac{dx}{dt} = v_0 + at.$$

In other words, 'velocity $= \dfrac{d}{dt}$ (position)'. In this section we will see how we can generalize this idea to three dimensions.

If f is a scalar function, then the derivative of f at t, also called the derivative of $f(t)$ with respect to t, is defined as

See the Calculus block of *M101*

$$\frac{df(t)}{dt} = \lim_{h \to 0} \left(\frac{f(t+h) - f(t)}{h} \right)$$

We define the **derivative of a vector function f** in the same way.

If $\mathbf{f}$ is a vector function, then its derivative at t, also called **the derivative of $\mathbf{f}(t)$ with respect to** t, is defined as

$$\frac{d\mathbf{f}(t)}{dt} = \lim_{h\to 0}\left(\frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h}\right).$$

The notation $\mathbf{f}'(t)$ can be used in place of $\frac{d\mathbf{f}(t)}{dt}$, just as with scalar functions.

Differentiation of vector functions does not always require the use of this definition. We shall develop rules for differentiating sums and products of vector functions using this definition and then apply these rules to combinations of vector functions.

Example 5

Show that the derivative of a constant vector is zero.

Solution

Suppose $\mathbf{f}(t) = \mathbf{c}$ is a vector which is constant in magnitude and direction, so that $\mathbf{f}(t+h) = \mathbf{c}$. Then

$$\mathbf{f}(t+h) - \mathbf{f}(t) = \mathbf{c} - \mathbf{c} = \mathbf{0}.$$

The definition of $\frac{d\mathbf{f}(t)}{dt}$ thus leads to the result that

$$\frac{d\mathbf{c}}{dt} = \lim_{h\to 0}\left(\frac{\mathbf{c}-\mathbf{c}}{h}\right) = \mathbf{0}.$$

Example 6

Show that, if a vector function $\mathbf{f}$ is defined by the scaling

$$\mathbf{f}(t) = g(t)\mathbf{c}$$

where g is a scalar function and $\mathbf{c}$ is a constant vector, then

$$\frac{d\mathbf{f}(t)}{dt} = \frac{dg(t)}{dt}\mathbf{c}.$$

Solution

We prove this result by applying the definition of $\frac{d\mathbf{f}(t)}{dt}$:

$$\frac{d\mathbf{f}(t)}{dt} = \lim_{h\to 0}\frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h},$$

which on substitution for $\mathbf{f}$ gives

$$\begin{aligned}\frac{d\mathbf{f}(t)}{dt} &= \lim_{h\to 0}\frac{g(t+h)\mathbf{c}-g(t)\mathbf{c}}{h}\\ &= \left(\lim_{h\to 0}\frac{g(t+h)-g(t)}{h}\right)\mathbf{c}\\ &= \frac{dg(t)}{dt}\mathbf{c}.\end{aligned}$$

Warning: remember that this result is only true if $\mathbf{c}$ *is a constant vector.*

Exercise 2

If $\mathbf{c}$ is a constant vector, find $\frac{d\mathbf{f}(t)}{dt}$ where $\mathbf{f}(t) = t^2e^t\mathbf{c}$.

Exercise 3

If $\mathbf{f}$ and $\mathbf{g}$ are vector functions of the scalar t, use the definition of differentiation to show that

$$\frac{d}{dt}(\mathbf{f}(t)+\mathbf{g}(t)) = \frac{d\mathbf{f}(t)}{dt} + \frac{d\mathbf{g}(t)}{dt}.$$

[*Solutions to Exercises 2 and 3 on p. 53*]

We can now put together the results of Example 6 and Exercise 3 above to establish the rule for differentiating a vector function given in terms of Cartesian components.

If the vector function $\mathbf{f}$ is given in Cartesian component form by

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$$

where f_1, f_2 and f_3 are real functions of the variable t, then

$$\boxed{\frac{d\mathbf{f}(t)}{dt} = \frac{df_1(t)}{dt}\mathbf{i} + \frac{df_2(t)}{dt}\mathbf{j} + \frac{df_3(t)}{dt}\mathbf{k}.}$$

Thus the derivative of the vector function is equal to the vector made up of the derivatives of the components of $\mathbf{f}(t)$. The reason for this simple expression is that the Cartesian unit vectors are constant vectors. This expression gives us an easy way of finding derivatives: we just differentiate functions that we are familiar with in the usual way. The following example illustrates this.

Example 7

Differentiate the following vector functions with respect to t:

(i) $\mathbf{f}(t) = (\cos 3t)\mathbf{i} + (2\sin 3t)\mathbf{j} + t^2\mathbf{k}$,

(ii) $\mathbf{g}(t) = (e^{-t}\cos t)\mathbf{i} + (e^{-t}\sin t)\mathbf{j}$.

Solution

(i) All we have to do is to differentiate the components of $\mathbf{f}$, i.e. the functions $\cos 3t$, $2\sin 3t$ and t^2. This should be familiar to you:

$$\frac{d}{dt}(\cos 3t) = -3\sin 3t,$$

$$\frac{d}{dt}(2\sin 3t) = 6\cos 3t,$$

$$\frac{d}{dt}(t^2) = 2t,$$

so that

$$\frac{d\mathbf{f}(t)}{dt} = (-3\sin 3t)\mathbf{i} + (6\cos 3t)\mathbf{j} + 2t\mathbf{k}.$$

(ii) In the same way we must differentiate the functions $e^{-t}\cos t$ and $e^{-t}\sin t$ to give

$$(-e^{-t}\cos t - e^{-t}\sin t) \quad \text{and} \quad (-e^{-t}\sin t + e^{-t}\cos t)$$

so that

$$\frac{d\mathbf{g}(t)}{dt} = -e^{-t}(\cos t + \sin t)\mathbf{i} + e^{-t}(-\sin t + \cos t)\mathbf{j}.$$

Now you have a go at the following exercise.

Exercise 4

Write down the derivatives of the following vector functions:

(i) $\mathbf{f}(t) = (at^2 + vt)\mathbf{i} + ut\mathbf{j} + gt^2\mathbf{k}$ where a, v, u and g are scalar constants;

(ii) $\mathbf{f}(t) = te^t\mathbf{i} + (\cos t)\mathbf{j} - e^{-2t}\mathbf{k}$.

[*Solution on p. 53*]

You have seen that the derivative of a constant vector is zero and the derivative of a sum of two vectors is the sum of the derivatives of the two vectors. These are the same rules as apply to derivatives of ordinary functions. Now let's have a look at the derivatives of products involving vectors.

Example 8

If the scalar m and the vectors $\mathbf{u}$ and $\mathbf{v}$ are functions of the scalar variable t, show that:

(i) $\dfrac{d}{dt}(m\mathbf{u}) = \dfrac{dm}{dt}\mathbf{u} + m\dfrac{d\mathbf{u}}{dt}$;

(ii) $\dfrac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\dfrac{d\mathbf{v}}{dt}$.

Solution

(i) In Example 6 we have already proved this result for a constant vector $\mathbf{c}$, i.e. in the present notation,

$$\frac{d}{dt}(m\mathbf{c}) = \frac{dm}{dt}\mathbf{c}.$$

To prove the result for a general vector function $\mathbf{u}(t)$, we write it in Cartesian component form:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}.$$

Then

$$m\mathbf{u} = (mu_1)\mathbf{i} + (mu_2)\mathbf{j} + (mu_3)\mathbf{k}.$$

Now to differentiate $m\mathbf{u}$ we simply differentiate the components mu_1, mu_2 and mu_3. These are just products of ordinary functions and so we use the rules of differentiation that we are familiar with, i.e.

$$\frac{d}{dt}(mu_i) = \frac{dm}{dt}u_i + m\frac{du_i}{dt} \quad \text{for } i = 1, 2, 3.$$

Thus

$$\frac{d}{dt}(m\mathbf{u}) = \left(\frac{dm}{dt}u_1 + m\frac{du_1}{dt}\right)\mathbf{i} + \left(\frac{dm}{dt}u_2 + m\frac{du_2}{dt}\right)\mathbf{j} + \left(\frac{dm}{dt}u_3 + m\frac{du_3}{dt}\right)\mathbf{k},$$

and collecting separately the terms involving $\dfrac{dm}{dt}$ and m the right hand side becomes

$$\frac{dm}{dt}(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) + m\left(\frac{du_1}{dt}\mathbf{i} + \frac{du_2}{dt}\mathbf{j} + \frac{du_3}{dt}\mathbf{k}\right),$$

which leads to the result required:

$$\frac{d}{dt}(m\mathbf{u}) = \frac{dm}{dt}\mathbf{u} + m\frac{d\mathbf{u}}{dt}.$$

(ii) To prove the result for the dot product, write $\mathbf{u}$ and $\mathbf{v}$ in Cartesian component form:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Then

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

so that

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3)\\ &= \frac{du_1}{dt}v_1 + u_1\frac{dv_1}{dt} + \frac{du_2}{dt}v_2 + u_2\frac{dv_2}{dt} + \frac{du_3}{dt}v_3 + u_3\frac{dv_3}{dt},\end{aligned}$$

and this simplifies to give the result

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}.$$

Now, you have a go at obtaining the rule for differentiating the cross product of two vector functions.

Exercise 5

If the vectors $\mathbf{u}$ and $\mathbf{v}$ are functions of the scalar t, show that

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \left(\frac{d\mathbf{u}}{dt} \times \mathbf{v}\right) + \left(\mathbf{u} \times \frac{d\mathbf{v}}{dt}\right).$$

[*Solution on p. 53*]

We have proved the rules for differentiating products of vectors by using the component form of the vectors. However, we can use the rules without referring to the components, as the following example shows.

Example 9

Differentiate the following expressions:

(i) $\mathbf{u} \cdot \mathbf{u}$,

(ii) $u^3\mathbf{u} + (\mathbf{a} \times \mathbf{u})$,

where $\mathbf{u}$ is a vector function of t, $u = |\mathbf{u}|$ and $\mathbf{a}$ is a constant vector.

Solution

(i)
$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} + \mathbf{u} \cdot \frac{d\mathbf{u}}{dt}$$
$$= 2\mathbf{u} \cdot \frac{d\mathbf{u}}{dt}.$$

(ii)
$$\frac{d}{dt}(u^3\mathbf{u}) = 3u^2\frac{du}{dt}\mathbf{u} + u^3\frac{d\mathbf{u}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{u}) = \left(\frac{d\mathbf{a}}{dt} \times \mathbf{u}\right) + \left(\mathbf{a} \times \frac{d\mathbf{u}}{dt}\right)$$
$$= \mathbf{a} \times \frac{d\mathbf{u}}{dt}, \text{ since } \frac{d\mathbf{a}}{dt} = \mathbf{0}.$$

Thus

$$\frac{d}{dt}\big(u^3\mathbf{u} + (\mathbf{a} \times \mathbf{u})\big) = 3u^2\frac{du}{dt}\mathbf{u} + u^3\frac{d\mathbf{u}}{dt} + \left(\mathbf{a} \times \frac{d\mathbf{u}}{dt}\right).$$

The first part of this example can be written in terms of scalar functions, because

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = u^2,$$

so that

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = \frac{d}{dt}(u^2) = 2u\frac{du}{dt}.$$

This leads to an interesting result for a vector of *constant* magnitude. If $\mathbf{e}$ is such a vector, then $|\mathbf{e}|$ = constant and so $\frac{d}{dt}|\mathbf{e}| = 0$. Thus,

$$\frac{d}{dt}(\mathbf{e} \cdot \mathbf{e}) = 2\mathbf{e} \cdot \frac{d\mathbf{e}}{dt} = \frac{d}{dt}|\mathbf{e}|^2 = 2|\mathbf{e}|\frac{d|\mathbf{e}|}{dt}$$
$$= 0.$$

Now remember that $\mathbf{e}$ is not necessarily a constant vector, because although its magnitude is constant the direction of $\mathbf{e}$ may vary. However, the equation

$$\mathbf{e} \cdot \frac{d\mathbf{e}}{dt} = 0$$

shows that for a vector $\mathbf{e}$ of *constant* magnitude, *its derivative* $\frac{d\mathbf{e}}{dt}$ *is always perpendicular to* $\mathbf{e}$.

This example again demonstrates the importance of the two properties defining a vector: (i) its magnitude and (ii) its direction. Even though one of these properties may be constant, the derivative of the vector is not necessarily equal to the zero vector.

Exercise 6

(i) If $\mathbf{u} = (3\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}$, then verify by calculation that $\dfrac{d\mathbf{u}}{dt}$ is perpendicular to $\mathbf{u}$.

(ii) If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, m is a constant scalar and $\mathbf{u} = (\cos mt)\mathbf{a} + (\sin mt)\mathbf{b}$, then show that

$$\mathbf{u} \times \frac{d\mathbf{u}}{dt} = m(\mathbf{a} \times \mathbf{b}).$$

(iii) If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, $\mathbf{r}$ is a vector which is a function of t and $r = |\mathbf{r}|$, then differentiate the expression

$$r^2\mathbf{r} + (\mathbf{a} \cdot \mathbf{r})\mathbf{b}.$$

[*Solution on p. 53*]

4.4 Second derivatives of vector functions

The derivative of the vector function $\mathbf{f}$, i.e. $\dfrac{d\mathbf{f}(t)}{dt}$, is again a function of t, and we could differentiate this new vector function. We define the **second derivative** of a vector function $\mathbf{f}$ by differentiating *twice* instead of once. That is, if $\mathbf{f}$ is a vector function with derivative $\dfrac{d\mathbf{f}(t)}{dt}$, then the derivative of *that* derivative is $\dfrac{d}{dt}\left(\dfrac{d\mathbf{f}(t)}{dt}\right)$, and we call it the **second derivative** of $\mathbf{f}$. For example, if

$$\mathbf{f}(t) = t^2\mathbf{i} + t^3\mathbf{j},$$

then the first derivative of $\mathbf{f}$ is

$$\frac{d\mathbf{f}(t)}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j}$$

and the second derivative of $\mathbf{f}$ is

$$\frac{d}{dt}\left(\frac{d\mathbf{f}(t)}{dt}\right) = 2\mathbf{i} + 6t\mathbf{j}.$$

We often write the second derivative as

$$\frac{d^2\mathbf{f}(t)}{dt^2}.$$

In terms of the Cartesian coordinates of $\mathbf{f}$, we have

$$\frac{d^2\mathbf{f}(t)}{dt^2} = \frac{d^2f_1(t)}{dt^2}\mathbf{i} + \frac{d^2f_2(t)}{dt^2}\mathbf{j} + \frac{d^2f_3(t)}{dt^2}\mathbf{k}.$$

Exercise 7

(i) If $\mathbf{f}(t) = (at^2 + vt)\mathbf{i} + ut\mathbf{j} + gt^2\mathbf{k}$, where a, v, u and g are scalar constants, find $\dfrac{d^2\mathbf{f}(t)}{dt^2}$.

(ii) If $\mathbf{r}(t) = (\cos mt)\mathbf{a} + (\sin mt)\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and m is a constant scalar, then show that

$$\frac{d^2\mathbf{r}}{dt^2} + m^2\mathbf{r} = \mathbf{0}.$$

[*Solution on p. 53*]

4.5 Geometrical interpretation

What does the derivative of a vector function mean geometrically? We know that a vector function can be visualized as a curve in space; so let us consider a particle moving along a curve. Consider the position vector $\mathbf{r}(t)$ of a particle P. Then as the time t varies the particle moves along a curve in space. Consider two positions of the particle, P and P' at times t and $t + h$; then $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$ are the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OP'}$. Now

$$\mathbf{r}(t + h) - \mathbf{r}(t) = \overrightarrow{OP'} - \overrightarrow{OP}$$

is the displacement vector of P' relative to P, i.e. $\overrightarrow{PP'}$, and h is a scalar, so that the quotient $\dfrac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h} = \dfrac{\overrightarrow{PP'}}{h}$ may be written as $\dfrac{1}{h}\,(\overrightarrow{PP'})$, i.e. a scaling of the vector

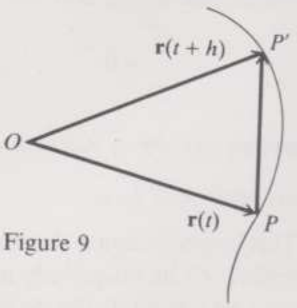


Figure 9

$\overrightarrow{PP'}$, so that $\frac{1}{h}(\overrightarrow{PP'})$ *is a vector*. As $h \to 0$ the point P' approaches the point P and the vector $\overrightarrow{PP'}$ becomes tangential to the curve at P. Then at P, $\frac{d\mathbf{r}}{dt}$ *is a vector which lies along the tangent to the curve at* P and is directed in the sense in which the particle at P would move if the value of t was increased. Now $\frac{d\mathbf{r}}{dt}$ is the limit of the ratio $\frac{\text{displacement}}{\text{time}}$, and if h is small the magnitude of the displacement, i.e. $|\overrightarrow{PP'}|$, approximates to the actual distance travelled by the particle in a time h. The ratio $\frac{\text{distance}}{\text{time}}$ is used to define average speed in *Unit 4*. We can now generalize the ideas of *Unit 4* so that the **velocity of a particle,** $\mathbf{v}$, is defined from the position vector $\mathbf{r}$ by

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt}}$$

and $\frac{d\mathbf{r}}{dt}$ is directed along the tangent to the curve along which the particle travels.

Example 10

Show that the velocity of a particle which travels round a circle is perpendicular to its position vector relative to the centre of the circle.

Solution

Suppose the particle has position vector $\mathbf{r}$. Then since the path of P is a circle, $|\mathbf{r}|$ is constant. But the derivative of a vector with constant magnitude is perpendicular to the vector, and so the velocity of the particle, $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, is perpendicular to $\mathbf{r}$.

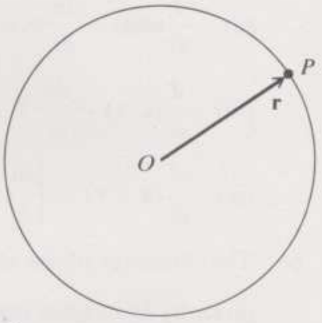


Figure 10

Exercise 8

A particle travels along the straight line with equation

$$\mathbf{r} = \mathbf{a} + mt\mathbf{d}$$

where $\mathbf{a}$ and $\mathbf{d}$ are constant vectors and m is a constant scalar. Show that the velocity of the particle is parallel to the direction of the line.

[*Solution on p. 54*]

The first and second derivatives of the position vector $\mathbf{r}$ have an important part to play in mechanics in three dimensions. You have seen in this section that $\frac{d\mathbf{r}}{dt}$ is the vector representing the velocity, and you will see in the next unit that $\frac{d^2\mathbf{r}}{dt^2}$ is the vector representing the acceleration. It certainly agrees with the result for one-dimensional motion; for if $\mathbf{r} = x\mathbf{i}$, then $\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i}$, and the acceleration $\frac{d^2x}{dt^2}$ is simply the second derivative of position, x.

Summary of Section 4

1. A **vector function f** is a function whose domain is the real numbers and whose codomain is the set of all three-dimensional vectors. The Cartesian components of $\mathbf{f}(t)$ are real functions of the scalar variable t. The magnitude and direction of $\mathbf{f}(t)$ are (possibly constant) functions of t.
2. A vector which is constant in both its magnitude and direction is called a **constant vector**. Examples of constant vectors are the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

3. The vector function $\mathbf{f}$ can be visualized as a curve in three dimensions by writing

$$\overrightarrow{OP} = \mathbf{f}(t).$$

The curve is the path of P as t varies, where O is a fixed origin and $\overrightarrow{OP}$ is the position vector of P.

4. The derivative of the vector function $\mathbf{f}$ at t is defined by

$$\frac{d\mathbf{f}(t)}{dt} = \lim_{h \to 0} \left(\frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h} \right).$$

The vector $\dfrac{d\mathbf{f}(t)}{dt}$ is a function of the variable t.

For a vector function given in the Cartesian component form

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}.$$

this reduces to

$$\frac{d\mathbf{f}(t)}{dt} = \frac{df_1(t)}{dt}\mathbf{i} + \frac{df_2(t)}{dt}\mathbf{j} + \frac{df_3(t)}{dt}\mathbf{k}.$$

5. The rules for the differentiation of sums and products of vectors are:

(i) $\dfrac{d}{dt}(\mathbf{u} + \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} + \dfrac{d\mathbf{v}}{dt}$;

(ii) $\dfrac{d}{dt}(m\mathbf{u}) = \dfrac{dm}{dt}\mathbf{u} + m\dfrac{d\mathbf{u}}{dt}$;

(iii) $\dfrac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \dfrac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \dfrac{d\mathbf{v}}{dt}$;

(iv) $\dfrac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \left(\dfrac{d\mathbf{u}}{dt} \times \mathbf{v}\right) + \left(\mathbf{u} \times \dfrac{d\mathbf{v}}{dt}\right)$.

6. The direction of the vector $\dfrac{d\mathbf{f}(t)}{dt}$ is along the tangent at $\mathbf{f}(t)$ to the curve given by the vector function $\mathbf{f}$.

7. If $|\mathbf{v}|$ is constant, then $\mathbf{v} \cdot \dfrac{d\mathbf{v}}{dt} = 0$.

End of section exercises

Exercise 9

The vector $\mathbf{r}$ is defined by

$$\mathbf{r} = r\mathbf{e}$$

where $\mathbf{e}$ is a unit vector and r represents the magnitude of $\mathbf{r}$, both $\mathbf{e}$ and r being functions of t. Show that

$$\frac{dr}{dt} = \mathbf{e} \cdot \frac{d\mathbf{r}}{dt}.$$

Exercise 10

If $\mathbf{u} = (\cos 2t)\mathbf{i} - t^2\mathbf{j} + (e^t \sin t)\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + t\mathbf{j} + (2e^{-t}\sin t)\mathbf{k}$, then calculate $\dfrac{d\mathbf{u}}{dt}$ and $\dfrac{d}{dt}(\mathbf{u} \cdot \mathbf{v})$.

(Hint: calculate $\mathbf{u} \cdot \mathbf{v}$ first using the trigonometric formula $\cos 2t = 1 - 2\sin^2 t$.)

Exercise 11

Find a unit vector along the direction of the velocity for a particle whose position vector is given by

$$\mathbf{r} = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

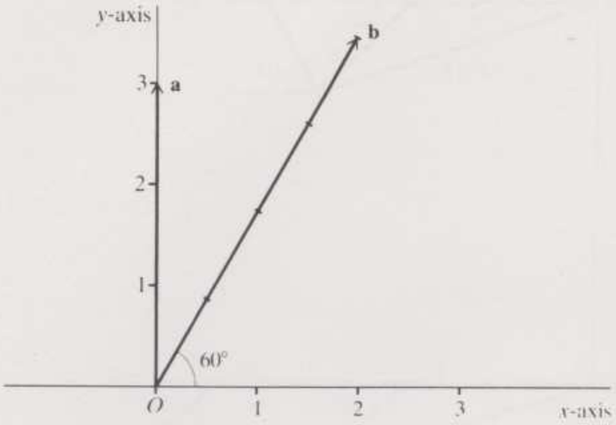
at the point given by $t = \pi/2$.

[*Solutions to Exercises 9–11 on p. 54*]

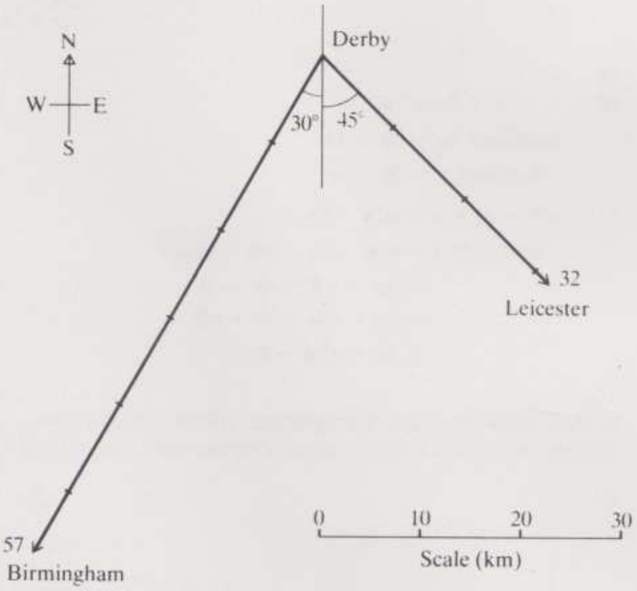
Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

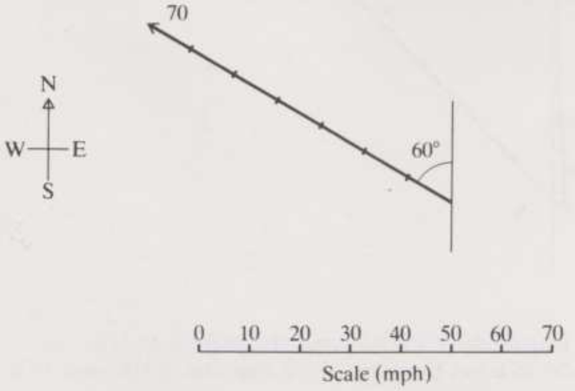
1.



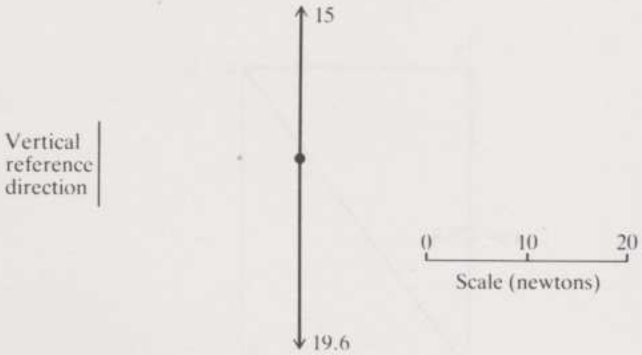
2.



3.



4.
The two forces acting on the stone are the weight, equal to $2 \times 9.8 = 19.6$ newtons vertically downwards and the air resistance, 15 newtons vertically upwards.



5.
$\mathbf{f} = \mathbf{b}$, as both are of length 2 units and both point in the positive y direction.

6.

(i) Leeds to Bristol: $-\mathbf{d}$
Leeds to Leeds: $\mathbf{0}$.

(ii) (a) $2\mathbf{v}$
(b) $-\mathbf{v}$
(c) $\mathbf{0}$.

(iii) The vector $-1.5\mathbf{v}$ has magnitude $1.5|\mathbf{v}|$ and direction opposite to $\mathbf{v}$. The vector $-k\mathbf{v}$ has magnitude $k|\mathbf{v}|$ and direction opposite to $\mathbf{v}$.

(iv) (a) The vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal in length, are parallel and point the same way (i.e. are equal in direction). Thus,

$$\overrightarrow{AB} = \overrightarrow{DC}.$$

(b) The vectors $\overrightarrow{BC}$ and $\overrightarrow{DA}$ are equal in length, are parallel but point in opposite directions. Thus,

$$\overrightarrow{BC} = -\overrightarrow{DA}.$$

(v) $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$ is a scaling of $\mathbf{a}$ by the scalar $m = \dfrac{1}{|\mathbf{a}|}$. The direction of $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$ is thus the same as that of $\mathbf{a}$, and the magnitude is $m|\mathbf{a}| = \dfrac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$.

7.
$\mathbf{a} = 2\mathbf{i}$, $\mathbf{b} = -2.5\mathbf{i}$, $\mathbf{c} = 3\mathbf{j}$, $\mathbf{d} = -\mathbf{j}$.

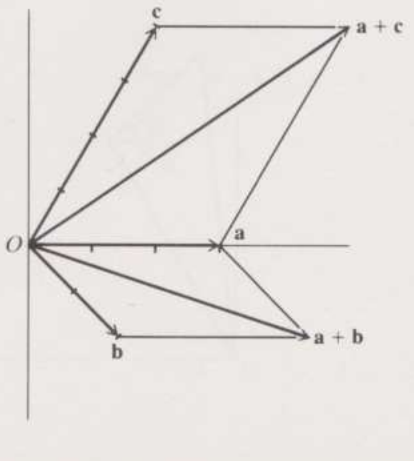
8.

(i) $-35\mathbf{j}$ (km/hr)

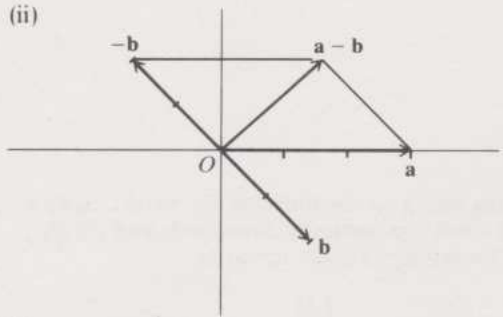
(ii) $-112\mathbf{i}$ (miles)

(iii) $112\mathbf{i}$ (miles).

9.
(i)



(ii)



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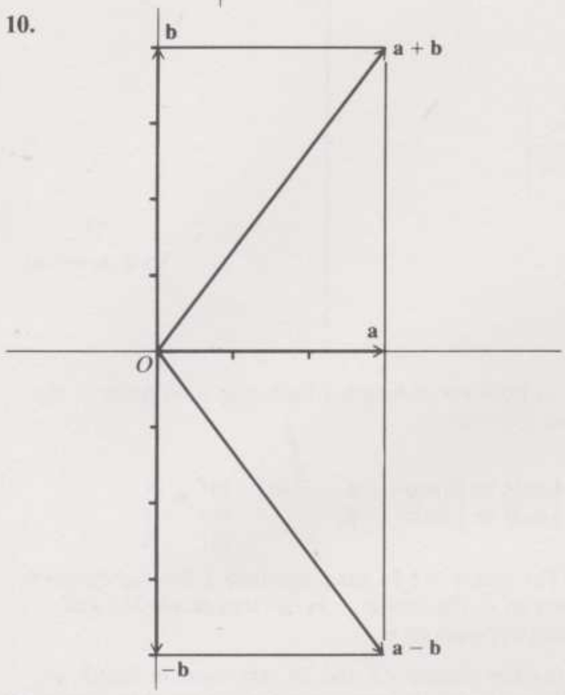
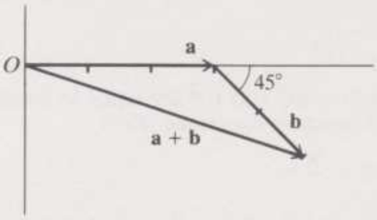
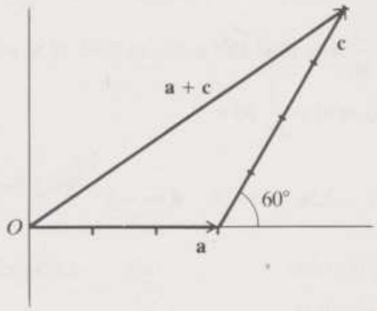
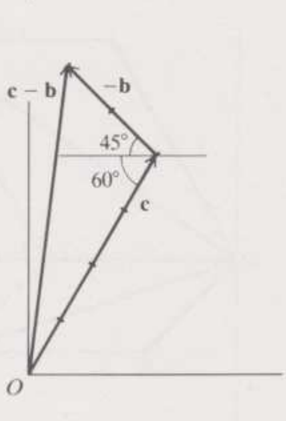


11.

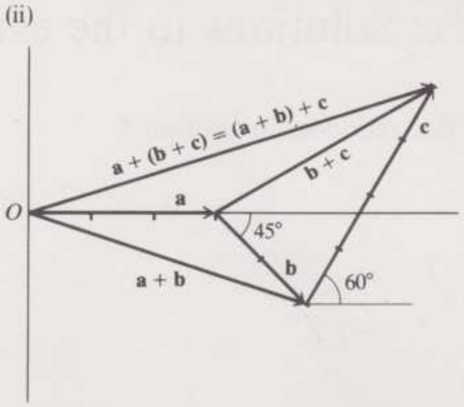
(i)

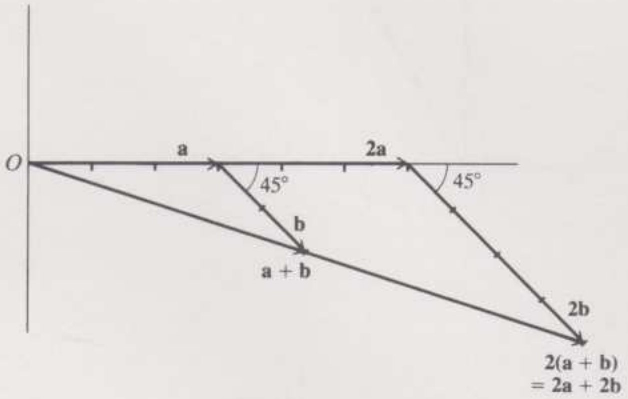






(ii)





12.

(i) $2\mathbf{b} + 4\mathbf{x} = 7\mathbf{a}$,

therefore $4\mathbf{x} = 7\mathbf{a} - 2\mathbf{b}$,

therefore $\mathbf{x} = \frac{7}{4}\mathbf{a} - \frac{1}{2}\mathbf{b}$.

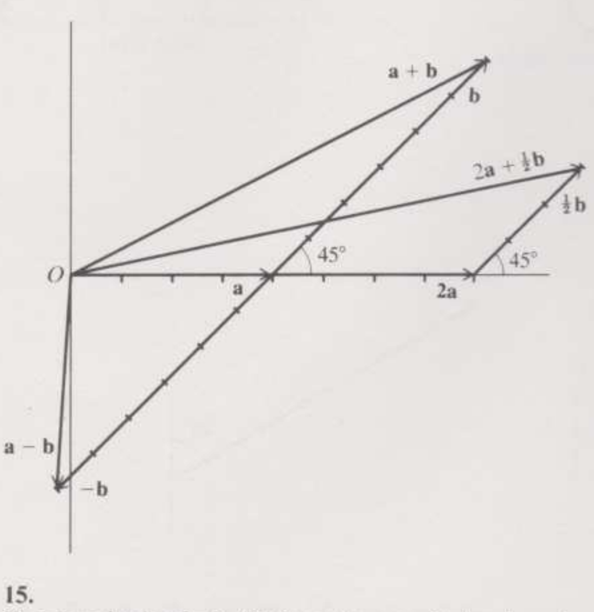
(ii) $n(\mathbf{b} - \mathbf{a}) + \mathbf{x} = m(\mathbf{a} - \mathbf{b})$,

$$\begin{aligned}\text{therefore } \mathbf{x} &= m(\mathbf{a} - \mathbf{b}) - n(\mathbf{b} - \mathbf{a})\\ &= m\mathbf{a} - m\mathbf{b} - n\mathbf{b} + n\mathbf{a}\\ &= (m + n)\mathbf{a} - (m + n)\mathbf{b}\\ &= (m + n)(\mathbf{a} - \mathbf{b}).\end{aligned}$$

13.

Scalar quantities: area, temperature, volume, energy, time.
Vector quantities: velocity, force, displacement, acceleration.

14.



15.

The magnitude of **v** is 4.7 times the magnitude of **u**. **v** is parallel to **u** but the sense of **v** is opposite to the sense of **u**.

16.

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (triangle rule)

$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$ (triangle rule)

Thus,

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}.$$

But $\overrightarrow{AD} = -\overrightarrow{DA}$, so we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = -\overrightarrow{DA},$$

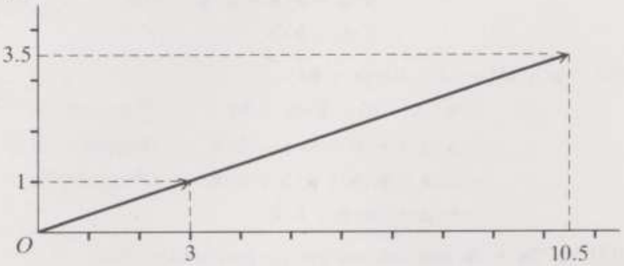
i.e. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0}.$

Solutions to the exercises in Section 2

1.

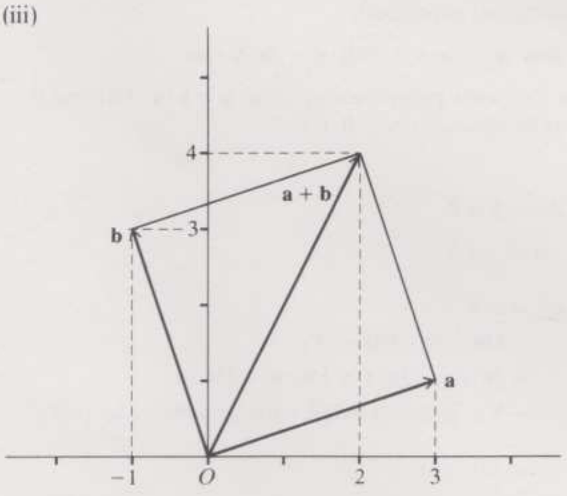
(i) $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$, $\mathbf{c} = -3\mathbf{i} - 2\mathbf{j}$, $\mathbf{d} = 3\mathbf{i} + \mathbf{j}$ $(= \mathbf{a})$.

(ii)



$3.5\mathbf{a} = 3.5\,(3\mathbf{i} + \mathbf{j}) = 10.5\mathbf{i} + 3.5\mathbf{j}$, as in the drawing.

(iii)



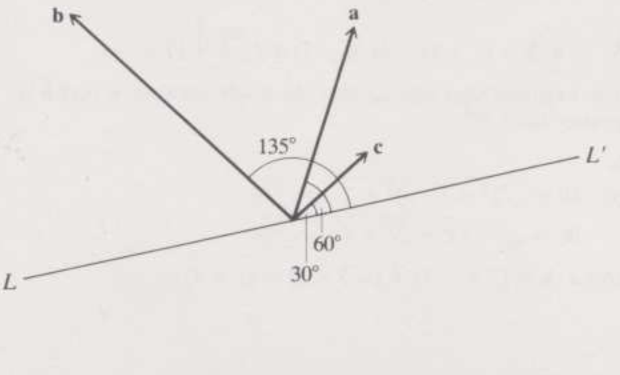
$$\mathbf{a} + \mathbf{b} = (3\mathbf{i} + \mathbf{j}) + (-\mathbf{i} + 3\mathbf{j})$$
$$= 2\mathbf{i} + 4\mathbf{j}, \text{ as in the drawing.}$$

(iv) $2\mathbf{a} + \mathbf{b} - \mathbf{c} = 2(3\mathbf{i} + \mathbf{j}) + (-\mathbf{i} + 3\mathbf{j}) - (-3\mathbf{i} - 2\mathbf{j})$
$= 8\mathbf{i} + 7\mathbf{j}.$

Thus $|2\mathbf{a} + \mathbf{b} - \mathbf{c}| = \sqrt{8^2 + 7^2} = \sqrt{113}.$

2.

(i)



(ii) First consider the vector $\mathbf{a}$.

The components of $\mathbf{a}$ are the numbers a_1 and a_2, which in terms of $|\mathbf{a}|$ and 60° are:

$$a_1 = |\mathbf{a}|\cos 60^\circ = 2\cos 60^\circ = 1;$$
$$a_2 = |\mathbf{a}|\sin 60^\circ = 2\sin 60^\circ = \sqrt{3}.$$

Thus,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} = \mathbf{i} + \sqrt{3}\,\mathbf{j}.$$

Similarly for $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{b} = (|\mathbf{b}|\cos 135^\circ)\mathbf{i} + (|\mathbf{b}|\sin 135^\circ)\mathbf{j}$$
$$= (3\cos 135^\circ)\mathbf{i} + (3\sin 135^\circ)\mathbf{j}$$
$$= -\frac{3}{\sqrt{2}}\,\mathbf{i} + \frac{3}{\sqrt{2}}\,\mathbf{j}.$$

$$\mathbf{c} = (|\mathbf{c}|\cos 30^\circ)\mathbf{i} + (|\mathbf{c}|\sin 30^\circ)\mathbf{j}$$
$$= (\cos 30^\circ)\mathbf{i} + (\sin 30^\circ)\mathbf{j}$$
$$= \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

We now have

$$\mathbf{a} + \mathbf{c} = (\mathbf{i} + \sqrt{3}\,\mathbf{j}) + \left(\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$$
$$= \left(1 + \frac{\sqrt{3}}{2}\right)\mathbf{i} + \left(\sqrt{3} + \frac{1}{2}\right)\mathbf{j}.$$

3.

Since $\mathbf{a} = \sqrt{3}\,\mathbf{i} - \mathbf{j}$, we have

$$|\mathbf{a}| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2,$$

and

$$\theta_{\mathbf{a}} = -\text{arc cos}\left(\frac{\sqrt{3}}{2}\right)$$

where we have chosen the negative sign because a_2 is negative.

Hence, $\theta_{\mathbf{a}} = -30^\circ$.

Since $\mathbf{b} = -3\mathbf{i} + 3\mathbf{j}$, we have

$$|\mathbf{b}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2},$$

and

$$\theta_{\mathbf{b}} = +\text{arc cos}\left(\frac{-3}{3\sqrt{2}}\right)$$

where this time we have chosen the positive sign because b_2 is positive.

Hence, $\theta_{\mathbf{b}} = 135^\circ$.

4.

Sets (ii), (iii) and (iv) are right-handed.

Either: try and twist your right hand so that the thumb represents Ox, the first finger Oy, then the middle finger is Oz and points the wrong way in (i) but the right way in (ii), (iii) and (iv);

or: imagine a screw turning from Ox to Oy, thus defining a direction Oz.

5.

(i) $\mathbf{d} = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) - 3(2\mathbf{i} - 3\mathbf{j} - \mathbf{k})$
$= -4\mathbf{i} + 11\mathbf{j} + 5\mathbf{k},$

$\mathbf{e} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - 2(2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) + 4(3\mathbf{i} + \mathbf{k})$
$= 9\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}.$

(ii) $|\mathbf{d}| = \sqrt{(-4)^2 + 11^2 + 5^2}$

$= \sqrt{162} = 9\sqrt{2},$

$|\mathbf{e}| = \sqrt{9^2 + 7^2 + 7^2}$

$= \sqrt{179}.$

(iii) $|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$

A unit vector in the direction of **a** is $\dfrac{1}{|\mathbf{a}|}\mathbf{a} = \dfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$

(iv) If $\mathbf{a} + \mathbf{x} = \mathbf{b},$

then
$$\begin{aligned} \mathbf{x} &= \mathbf{b} - \mathbf{a}, \\ &= (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \mathbf{i} - 4\mathbf{j} - 2\mathbf{k}. \end{aligned}$$

Thus the components of **x** are 1, -4 and -2.

6.

(i) $|\mathbf{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5},$

$|\mathbf{b}| = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}.$

The vector **a** lies in the *xy*-plane, and the angle θ which it makes with the *x*-axis is given by

$$\theta = -\arccos\left(\frac{2}{\sqrt{5}}\right),$$

where we have chosen the minus sign because a_2 is negative. Thus,

$$\theta \simeq -26.6°,$$

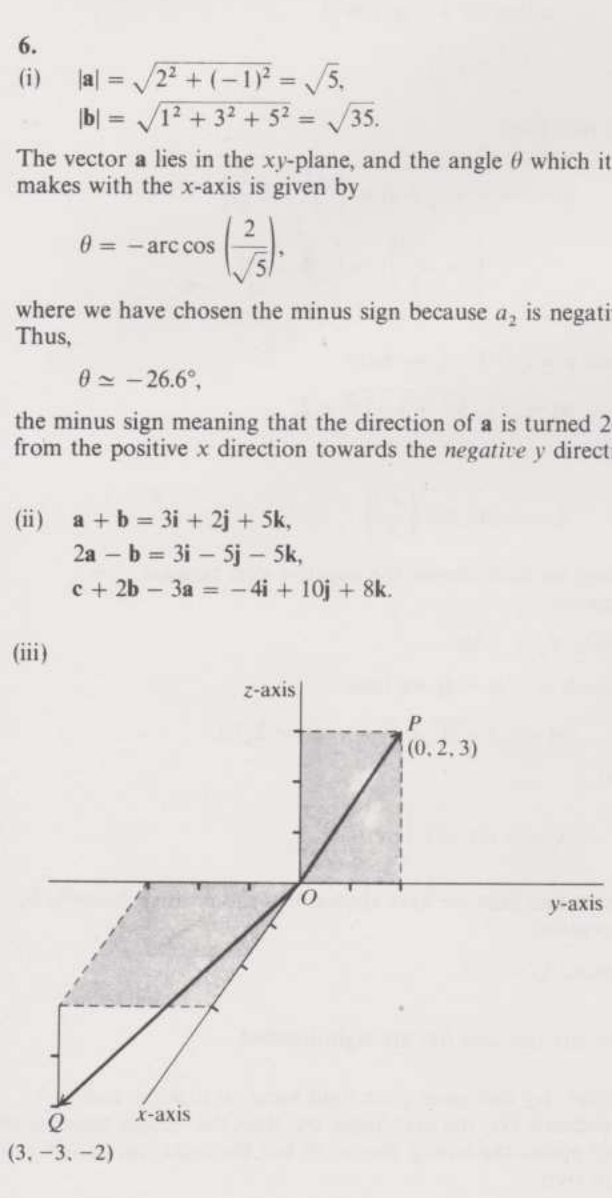
the minus sign meaning that the direction of **a** is turned 26.6° from the positive x direction towards the *negative y* direction.

(ii) $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k},$

$2\mathbf{a} - \mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k},$

$\mathbf{c} + 2\mathbf{b} - 3\mathbf{a} = -4\mathbf{i} + 10\mathbf{j} + 8\mathbf{k}.$

(iii)



The vector $\overrightarrow{PQ}$ is parallel to $2\mathbf{a} - \mathbf{b}$. The point Q is the end of the vector $\overrightarrow{OQ}$, which is given by

$$\begin{aligned} \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\ &= (2\mathbf{j} + 3\mathbf{k}) + (3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}) \\ &= 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}, \end{aligned}$$

so that Q is the point $(3, -3, -2)$.

Solutions to the exercises in Section 3

1. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = 2 \times 4 \times \cos 60° = 4.$

$\mathbf{a}\cdot\mathbf{c} = |\mathbf{a}|\,|\mathbf{c}|\cos\theta = 2 \times 1 \times \cos 90° = 0.$

$\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}|\,|\mathbf{c}|\cos\theta = 4 \times 1 \times \cos 30° = 2\sqrt{3}.$

$\mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|\,|\mathbf{b}|\cos\theta = 4 \times 4 \times \cos 0° = 16.$

2.
$$\begin{aligned} (\mathbf{a} + \mathbf{b})\cdot\mathbf{c} &= \mathbf{c}\cdot(\mathbf{a} + \mathbf{b}) && \text{(Property 1)} \\ &= \mathbf{c}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{b} && \text{(Property 2)} \\ &= \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c} && \text{(Property 1).} \end{aligned}$$

3.

(i)
$$\begin{aligned} (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) &= \mathbf{a}\cdot(\mathbf{a} - \mathbf{b}) + \mathbf{b}\cdot(\mathbf{a} - \mathbf{b}) && \text{(Exercise 2)} \\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} && \text{(Property 2)} \\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{b} && \text{(Property 1)} \\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} \end{aligned}$$

(ii)
$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a}\cdot(\mathbf{a} + \mathbf{b}) + \mathbf{b}\cdot(\mathbf{a} + \mathbf{b}) && \text{(Exercise 2)} \\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} && \text{(Property 2)} \\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} && \text{(Property 1)} \\ &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \end{aligned}$$

(iii) If $2\mathbf{a} + 3\mathbf{b}$ and $m\mathbf{a} + \mathbf{b}$ are perpendicular, then

$$(2\mathbf{a} + 3\mathbf{b})\cdot(m\mathbf{a} + \mathbf{b}) = 0.$$

Expanding this expression,

$$2m\mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + 3m\mathbf{b}\cdot\mathbf{a} + 3\mathbf{b}\cdot\mathbf{b} = 0.$$

Now **a** and **b** are perpendicular, so $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = 0$; and they are unit vectors, so $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = 1.$

Thus

$$2m + 3 = 0,$$

i.e. $m = -1.5.$

(iv)
$$\begin{aligned} |\mathbf{c}|^2 &= \mathbf{c}\cdot\mathbf{c} \\ &= (3\mathbf{u} + 5\mathbf{v})\cdot(3\mathbf{u} + 5\mathbf{v}) \\ &= 9\mathbf{u}\cdot\mathbf{u} + 15\mathbf{u}\cdot\mathbf{v} + 15\mathbf{v}\cdot\mathbf{u} + 25\mathbf{v}\cdot\mathbf{v} \\ &= 9 + 25 \text{ (since } \mathbf{u} \text{ and } \mathbf{v} \text{ are perpendicular unit vectors)} \\ &= 34. \end{aligned}$$

Thus, $|\mathbf{c}| = \sqrt{34}.$

4.
$$\begin{aligned} \mathbf{a}\cdot\mathbf{i} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot\mathbf{i} \\ &= a_1\mathbf{i}\cdot\mathbf{i} + a_2\mathbf{j}\cdot\mathbf{i} + a_3\mathbf{k}\cdot\mathbf{i} && \text{(Exercise 2)} \\ &= a_1 \text{ (since } i, \mathbf{j} \text{ and } \mathbf{k} \text{ are perpendicular unit vectors).} \end{aligned}$$

Similarly,

$$\mathbf{a}\cdot\mathbf{j} = a_2 \quad \text{and} \quad \mathbf{a}\cdot\mathbf{k} = a_3.$$

5. $\mathbf{a}\cdot\mathbf{b} = (4 \times 1) + (1 \times -3) + (-5 \times 1) = -4.$

The negative sign tells us that the angle between **a** and **b** is greater than 90°.

6.

(i) $|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14},$

$|\mathbf{b}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21},$

and $\mathbf{a}\cdot\mathbf{b} = (2 \times -1) + (-3 \times 2) + (1 \times 4) = -4.$

If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{-4}{\sqrt{14}\times\sqrt{21}} = \frac{-4}{7\sqrt{6}}.$$

The negative sign means that $\theta > 90°$, i.e.

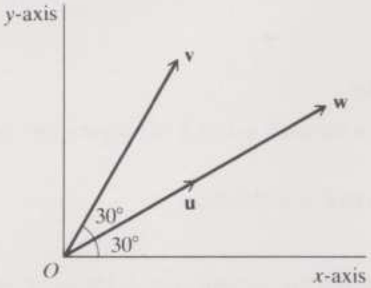
$$\theta \simeq 103.5°.$$

(ii) $\mathbf{a}\cdot\mathbf{c} = -2$, $\mathbf{a}\cdot\mathbf{d} = -3$, $\mathbf{a}\cdot\mathbf{e} = 0$.
Thus **e** is perpendicular to **a**.

(iii) A suitable vector along the line joining the origin to the point $(1, 1, 1)$ is $\mathbf{i}+\mathbf{j}+\mathbf{k}$. Thus, by the formula given at the end of Subsection 3.2, the projection of $\mathbf{a}+2\mathbf{b}$ onto this line is

$$\frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{a}+2\mathbf{b})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|} = \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{j}+9\mathbf{k})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|} = \frac{10}{\sqrt{3}}.$$

7.
For the sake of clarity, here is a diagram of **u**, **v** and **w**.



The cross products are all perpendicular to the xy-plane. A unit vector in the direction of $\mathbf{u}\times\mathbf{v}$ is **k**, and so

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (|\mathbf{u}|\,|\mathbf{v}|\sin 30°)\mathbf{k}\\ &= (2\times 3\times\tfrac{1}{2})\mathbf{k}\\ &= 3\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\mathbf{u}\times\mathbf{w} &= (|\mathbf{u}|\,|\mathbf{w}|\sin 0°)\mathbf{k}\\ &= \mathbf{0}, \text{ as } \sin 0° = 0.\end{aligned}$$

A unit vector in the direction of $\mathbf{v}\times\mathbf{w}$ is $-\mathbf{k}$, and so

$$\begin{aligned}\mathbf{v}\times\mathbf{w} &= (|\mathbf{v}|\,|\mathbf{w}|\sin 30°)(-\mathbf{k})\\ &= (3\times 4\times\tfrac{1}{2})(-\mathbf{k})\\ &= -6\mathbf{k}.\end{aligned}$$

8.
(i)
$$\begin{aligned}(\mathbf{a}+\mathbf{b}+\mathbf{c})\times(\mathbf{b}-\mathbf{c}) &= ((\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{b}) - ((\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{c})\\ &= ((\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{b})+(\mathbf{c}\times\mathbf{b}))\\ &\quad - ((\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{c}))\\ &= (\mathbf{a}\times\mathbf{b})+\mathbf{0}-(\mathbf{b}\times\mathbf{c})-(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\times\mathbf{c})-\mathbf{0}\\ &= (\mathbf{a}\times\mathbf{b})-2(\mathbf{b}\times\mathbf{c})-(\mathbf{a}\times\mathbf{c}).\end{aligned}$$

(ii)
$$\begin{aligned}\mathbf{d}\times\mathbf{e} &= (2\mathbf{u}-\mathbf{v})\times(\mathbf{u}+2\mathbf{v})\\ &= ((2\mathbf{u}-\mathbf{v})\times\mathbf{u}) + ((2\mathbf{u}-\mathbf{v})\times(2\mathbf{v}))\\ &= (2(\mathbf{u}\times\mathbf{u})-(\mathbf{v}\times\mathbf{u})) + (4(\mathbf{u}\times\mathbf{v})-2(\mathbf{v}\times\mathbf{v}))\\ &= \mathbf{0}+(\mathbf{u}\times\mathbf{v})+4(\mathbf{u}\times\mathbf{v})-\mathbf{0}\\ &= 5(\mathbf{u}\times\mathbf{v}).\end{aligned}$$

9.
(i) $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$.

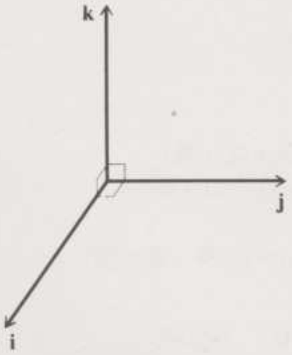
(ii)
$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{b}) &= \mathbf{a}\times\mathbf{0}\\ &= \mathbf{0}.\end{aligned}$$

However, $(\mathbf{a}\times\mathbf{b})\times\mathbf{b}$ is not **0**. This is because **a** and **b** are not parallel, so that $\mathbf{a}\times\mathbf{b}$ is a non-zero vector. Being perpendicular to **a** and to **b**, $\mathbf{a}\times\mathbf{b}$ is certainly not parallel to **b**, and so

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{b} \neq \mathbf{0}, \text{ i.e.}$$
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{b} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{b}).$$

10.
(i) **i**, **j** and **k** are unit vectors forming a right-handed system. Thus, using the definition of the cross product we get

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\quad \mathbf{j}\times\mathbf{k}=\mathbf{i}\quad\text{and}\quad \mathbf{k}\times\mathbf{i}=\mathbf{j}.$$



(ii) Since $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ for any vector, we have

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}.$$

(iii)
$$\begin{aligned}(\mathbf{i}+\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) &= (\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}))\\ &\quad + (\mathbf{k}\times(\mathbf{i}+\mathbf{j}+\mathbf{k}))\\ &= (\mathbf{0}+\mathbf{k}+(-\mathbf{j})) + (\mathbf{j}+(-\mathbf{i})+\mathbf{0})\\ &= -\mathbf{i}+\mathbf{k}.\end{aligned}$$

$$\begin{aligned}(\mathbf{i}\times(\mathbf{i}+\mathbf{k})) - ((\mathbf{i}+\mathbf{j})\times\mathbf{k}) &= (\mathbf{0}+(-\mathbf{j})) - (-\mathbf{j}+\mathbf{i})\\ &= -\mathbf{i}.\end{aligned}$$

11.
$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (2\mathbf{i}-3\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+2\mathbf{j}+4\mathbf{k})\\ &= ((-3\times 4)-(1\times 2))\mathbf{i} + ((1\times -1)-(2\times 4))\mathbf{j}\\ &\quad + ((2\times 2)-(-3\times -1))\mathbf{k}\\ &= -14\mathbf{i}-9\mathbf{j}+\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{c} &= (2\mathbf{i}-3\mathbf{j}+\mathbf{k})\times(-4\mathbf{i}+6\mathbf{j}-2\mathbf{k})\\ &= ((-3\times -2)-(1\times 6))\mathbf{i} + ((1\times -4)-(2\times -2))\mathbf{j}\\ &\quad + ((2\times 6)-(-3\times -4))\mathbf{k}\\ &= \mathbf{0}.\end{aligned}$$

$$\begin{aligned}\mathbf{b}\times\mathbf{c} &= (-\mathbf{i}+2\mathbf{j}+4\mathbf{k})\times(-4\mathbf{i}+6\mathbf{j}-2\mathbf{k})\\ &= ((2\times -2)-(4\times 6))\mathbf{i} + ((4\times -4)-(-1\times -2))\mathbf{j}\\ &\quad + ((-1\times 6)-(2\times -4))\mathbf{k}\\ &= -28\mathbf{i}-18\mathbf{j}+2\mathbf{k}.\end{aligned}$$

Since $\mathbf{a}\times\mathbf{c}=\mathbf{0}$, the vectors **a** and **c** are parallel. In fact, $\mathbf{c} = -2\mathbf{a}$.

12. A vector perpendicular to **a** and **b** is $\mathbf{a}\times\mathbf{b}$; thus a unit vector in this direction is

$$\begin{aligned}\frac{1}{|\mathbf{a}\times\mathbf{b}|}(\mathbf{a}\times\mathbf{b}) &= \frac{1}{|\mathbf{a}\times\mathbf{b}|}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(4\mathbf{i}+4\mathbf{j}-7\mathbf{k})\\ &= \frac{1}{|\mathbf{a}\times\mathbf{b}|}(-18\mathbf{i}+18\mathbf{j})\\ &= \frac{-18\mathbf{i}+18\mathbf{j}}{|-18\mathbf{i}+18\mathbf{j}|}\\ &= \frac{-18\mathbf{i}+18\mathbf{j}}{18\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}).\end{aligned}$$

13.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (3\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\\ &= (3\times 2)+(2\times 1)+(1\times 3)=11.\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (3\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\\ &= 5\mathbf{i}-7\mathbf{j}-\mathbf{k}.\end{aligned}$$

The essential difference is that $\mathbf{a}\cdot\mathbf{b}$ is a scalar but $\mathbf{a}\times\mathbf{b}$ is a vector.

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) &= (3\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(5\mathbf{i}-7\mathbf{j}-\mathbf{k})\\ &= (3\times 5)+(2\times -7)+(1\times -1)=0.\end{aligned}$$

14.

$$\begin{aligned}\mathbf{b}\times\mathbf{c} &= (2\mathbf{j}+\mathbf{k})\times(2\mathbf{i}-\mathbf{k})\\ &= -2\mathbf{i}+2\mathbf{j}-4\mathbf{k},\end{aligned}$$

and

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (-\mathbf{i}+\mathbf{j})\cdot(-2\mathbf{i}+2\mathbf{j}-4\mathbf{k})\\ &= (-1\times -2)+(1\times 2)+0=4.\end{aligned}$$

15.

$$\begin{aligned}\mathbf{d}_1\times\mathbf{d}_2 &= (-5\mathbf{i}+2\mathbf{j}+4\mathbf{k})\times(-3\mathbf{i}-2\mathbf{j}+4\mathbf{k})\\ &= 16\mathbf{i}+8\mathbf{j}+16\mathbf{k}.\end{aligned}$$

$$\begin{aligned}|\mathbf{d}_1\times\mathbf{d}_2| &= \sqrt{16^2+8^2+16^2}\\ &= 8\sqrt{2^2+1^2+2^2}\\ &= 24.\end{aligned}$$

Thus,

$$\frac{\mathbf{d}_1\times\mathbf{d}_2}{|\mathbf{d}_1\times\mathbf{d}_2|}=\frac{16\mathbf{i}+8\mathbf{j}+16\mathbf{k}}{24}=\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k}),$$

and

$$\begin{aligned}\mathbf{a}\cdot\frac{\mathbf{d}_1\times\mathbf{d}_2}{|\mathbf{d}_1\times\mathbf{d}_2|} &= (8\mathbf{i}-4\mathbf{j})\cdot\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})\\ &= \frac{1}{3}\big((8\times 2)+(-4\times 1)+0\big)\\ &= 4.\end{aligned}$$

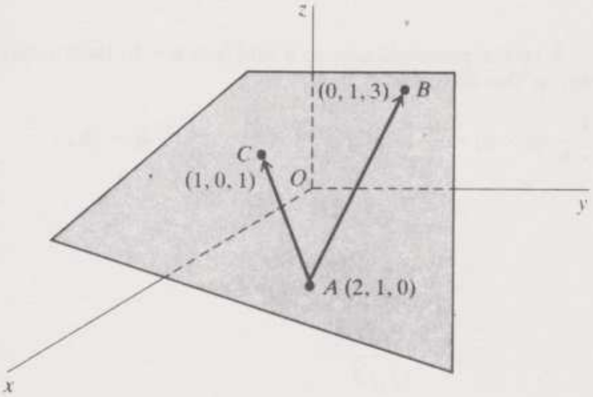
16.

(i) Two vectors lying in the plane containing A, B and C are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Now,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB}-\overrightarrow{OA}\\ &= (\mathbf{j}+3\mathbf{k})-(2\mathbf{i}+\mathbf{j})\\ &= -2\mathbf{i}+3\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{OC}-\overrightarrow{OA}\\ &= (\mathbf{i}+\mathbf{k})-(2\mathbf{i}+\mathbf{j})\\ &= -\mathbf{i}-\mathbf{j}+\mathbf{k}.\end{aligned}$$



(ii) A vector perpendicular to the plane is perpendicular to both the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Hence a normal vector is given by

$$\begin{aligned}\mathbf{n} &= \overrightarrow{AB}\times\overrightarrow{AC}\\ &= (-2\mathbf{i}+3\mathbf{k})\times(-\mathbf{i}-\mathbf{j}+\mathbf{k})\\ &= 3\mathbf{i}-\mathbf{j}+2\mathbf{k}.\end{aligned}$$

17.

The general equation for a plane with normal vector $\mathbf{n}=n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$ passing through the point (a,b,c) is given as

$$n_1(x-a)+n_2(y-b)+n_3(z-c)=0,$$

which can be rewritten as

$$n_1x+n_2y+n_3z=n_1a+n_2b+n_3c=\text{constant},$$

the form given in the exercise. We thus have

$$n_1=1,\ n_2=-2,\ n_3=5,$$

so that a vector perpendicular to the plane is

$$\mathbf{n}=\mathbf{i}-2\mathbf{j}+5\mathbf{k}.$$

Thus a unit vector perpendicular to the plane is

$$\frac{\mathbf{n}}{|\mathbf{n}|}=\frac{\mathbf{i}-2\mathbf{j}+5\mathbf{k}}{\sqrt{30}}.$$

18.

(i) No, it is a scalar.

(ii) Yes, if the angle between **a** and **b** is between 90° and 180°.

(iii) If $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta=|\mathbf{a}|\,|\mathbf{b}|$, then

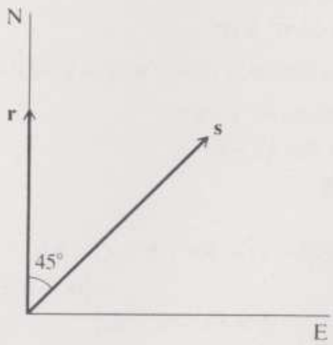
$$\cos\theta=1,$$

and so $\theta=0$, i.e. **a** and **b** are parallel and in the same sense.

(iv) If $\mathbf{a}\cdot\mathbf{b}=0$, then either $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$ or **a** and **b** are perpendicular.

(v) If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, then either $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$ or **a** and **b** are parallel.

19.



(i) **t** is perpendicular to both **r** and **s**, and its sense is down, i.e. 'into the page'. Conversely, the sense of $\mathbf{s}\times\mathbf{r}$ is up, i.e. 'out of the page'.

(ii) $\mathbf{t}\times\mathbf{r}$ is perpendicular to **t** (and thus in the horizontal plane), and perpendicular to **r**, and by the right hand rule or the screw rule its sense is due East.

(iii) $|\mathbf{t}|=|\mathbf{r}|\,|\mathbf{s}|\sin 45^\circ=\dfrac{1}{\sqrt{2}}$.

(iv) $\mathbf{t}\times(\mathbf{r}\times\mathbf{s})=\mathbf{t}\times\mathbf{t}=\mathbf{0}$.

(v) $\mathbf{r}\cdot\mathbf{s}=|\mathbf{r}|\,|\mathbf{s}|\cos 45^\circ=\dfrac{1}{\sqrt{2}}$.

(vi)
$$\begin{aligned}\mathbf{s}\cdot(\mathbf{t}\times\mathbf{r}) &= |\mathbf{s}|\,|\mathbf{t}\times\mathbf{r}|\cos 45^\circ \text{ (by part (ii))}\\ &= |\mathbf{s}|(|\mathbf{t}|\,|\mathbf{r}|\sin 90^\circ)\cos 45^\circ\\ &= 1\times\frac{1}{\sqrt{2}}\times 1\times 1\times\frac{1}{\sqrt{2}}=\frac{1}{2}.\end{aligned}$$

20.
$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\mathbf{0}$ for any non-zero vectors **a** and **b**, because $\mathbf{a}\times\mathbf{b}$ is perpendicular to **a** and the dot product of perpendicular vectors is zero.

(If **a** and/or **b** is the zero vector, then of course the product is still zero.)

Solutions to the exercises in Section 4

1.
The solution is given in the table below. All the vectors are two-dimensional, so that the direction of a vector is defined by the angle θ that the vector makes with the positive x direction.

vector	magnitude	whether constant	direction	whether constant
a	$1+t^2$	no	$\theta=0$	yes
b	1	yes	$\theta=t$	no
c	5	yes	$\theta=\text{arc cos}\left(\frac{4}{5}\right)$	yes
d	$\sqrt{5}\,e^t$	no	$\theta=\text{arc cos}\left(\frac{1}{\sqrt{5}}\right)$	yes

2. $\mathbf{f}(t)=t^2e^t\mathbf{c}$, so
$$\begin{aligned}\frac{d\mathbf{f}(t)}{dt}&=(2te^t+t^2e^t)\mathbf{c}\\ &=(2t+t^2)e^t\mathbf{c}.\end{aligned}$$

3.
$$\begin{aligned}\frac{d}{dt}(\mathbf{f}(t)+\mathbf{g}(t))&=\lim_{h\to 0}\frac{(\mathbf{f}(t+h)+\mathbf{g}(t+h))-(\mathbf{f}(t)+\mathbf{g}(t))}{h}\\ &=\lim_{h\to 0}\left(\frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h}+\frac{\mathbf{g}(t+h)-\mathbf{g}(t)}{h}\right)\\ &=\frac{d\mathbf{f}(t)}{dt}+\frac{d\mathbf{g}(t)}{dt}.\end{aligned}$$

4.

(i) $$\frac{d\mathbf{f}(t)}{dt}=(2at+v)\mathbf{i}+u\mathbf{j}+2gt\mathbf{k}.$$

(ii) $$\begin{aligned}\frac{d\mathbf{f}(t)}{dt}&=(te^t+e^t)\mathbf{i}+(-\sin t)\mathbf{j}-(-2e^{-2t})\mathbf{k}\\ &=(t+1)e^t\;\mathbf{i}-(\sin t)\mathbf{j}+2e^{-2t}\;\mathbf{k}.\end{aligned}$$

5.
Write **u** and **v** in component form:
$$\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}\quad\text{and}\quad\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}.$$
Then the **i** component of $\mathbf{u}\times\mathbf{v}$ is $u_2v_3-u_3v_2$. If we differentiate this, we get
$$\frac{du_2}{dt}v_3+u_2\frac{dv_3}{dt}-\frac{du_3}{dt}v_2-u_3\frac{dv_2}{dt},$$
which can be written as
$$\left(\frac{du_2}{dt}v_3-\frac{du_3}{dt}v_2\right)+\left(u_2\frac{dv_3}{dt}-u_3\frac{dv_2}{dt}\right).$$
The first term is the **i** component of $\frac{d\mathbf{u}}{dt}\times\mathbf{v}$ and the second term is the **i** component of $\mathbf{u}\times\frac{d\mathbf{v}}{dt}$.

Thus the **i** component of $\frac{d}{dt}(\mathbf{u}\times\mathbf{v})$ is equal to the **i** component of $\left(\frac{d\mathbf{u}}{dt}\times\mathbf{v}\right)+\left(\mathbf{u}\times\frac{d\mathbf{v}}{dt}\right)$.

We can do exactly the same for the **j** and **k** components, and this shows that
$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v})=\left(\frac{d\mathbf{u}}{dt}\times\mathbf{v}\right)+\left(\mathbf{u}\times\frac{d\mathbf{v}}{dt}\right).$$

6.

(i) $\frac{d\mathbf{u}}{dt}=-(6\sin 2t)\mathbf{i}+(6\cos 2t)\mathbf{j}$. Thus,
$$\begin{aligned}\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}&=-18\cos 2t\sin 2t+18\sin 2t\cos 2t\\ &=0.\end{aligned}$$

Neither **u** nor $\frac{d\mathbf{u}}{dt}$ is zero (in fact, $|\mathbf{u}|=3=\text{constant}$ and $\left|\frac{d\mathbf{u}}{dt}\right|=6=\text{constant}$). Therefore they are perpendicular.

(ii) $\frac{d\mathbf{u}}{dt}=-(m\sin mt)\mathbf{a}+(m\cos mt)\mathbf{b}$, since $\frac{d\mathbf{a}}{dt}=\frac{d\mathbf{b}}{dt}=\mathbf{0}$.
$$\begin{aligned}\mathbf{u}\times\frac{d\mathbf{u}}{dt}&=((\cos mt)\mathbf{a}+(\sin mt)\mathbf{b})\times(-(m\sin mt)\mathbf{a}\\ &\quad+(m\cos mt)\mathbf{b})\\ &=(-m\cos mt\sin mt)(\mathbf{a}\times\mathbf{a})+(m\cos^2 mt)(\mathbf{a}\times\mathbf{b})\\ &\quad-(m\sin^2 mt)(\mathbf{b}\times\mathbf{a})+(m\sin mt\cos mt)(\mathbf{b}\times\mathbf{b}).\end{aligned}$$
Now from the properties of the cross product,
$$\mathbf{a}\times\mathbf{a}=\mathbf{b}\times\mathbf{b}=\mathbf{0}\quad\text{and}\quad\mathbf{b}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{b}),$$
so this reduces to
$$\begin{aligned}\mathbf{u}\times\frac{d\mathbf{u}}{dt}&=(m\cos^2 mt+m\sin^2 mt)(\mathbf{a}\times\mathbf{b})\\ &=m(\mathbf{a}\times\mathbf{b}).\end{aligned}$$

(iii) $$\begin{aligned}&\frac{d}{dt}(r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\mathbf{b})\\ &=\left(\frac{d}{dt}(r^2)\right)\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\frac{d}{dt}(\mathbf{a}\cdot\mathbf{r})\right)\mathbf{b}+(\mathbf{a}\cdot\mathbf{r})\frac{d\mathbf{b}}{dt}\\ &=2r\frac{dr}{dt}\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\frac{d\mathbf{a}}{dt}\cdot\mathbf{r}+\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b}+(\mathbf{a}\cdot\mathbf{r})\frac{d\mathbf{b}}{dt}.\end{aligned}$$

But **a** and **b** are constant vectors, so $\frac{d\mathbf{a}}{dt}=\frac{d\mathbf{b}}{dt}=\mathbf{0}$. Thus,
$$\frac{d}{dt}(r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\mathbf{b})=2r\frac{dr}{dt}\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b}.$$

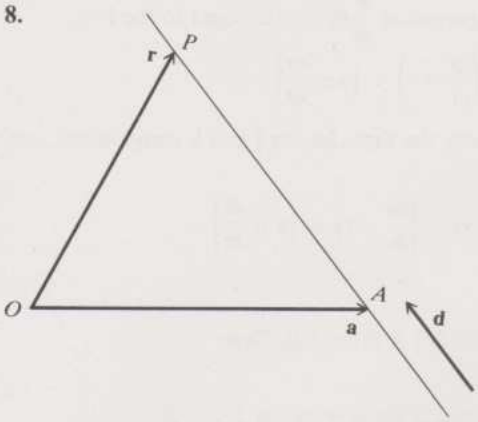
7.

(i) $$\begin{aligned}\frac{d\mathbf{f}(t)}{dt}&=(2at+v)\mathbf{i}+u\mathbf{j}+2gt\mathbf{k},\\ \frac{d^2\mathbf{f}(t)}{dt^2}&=2a\mathbf{i}+0\mathbf{j}+2g\mathbf{k}.\end{aligned}$$

(ii) $$\begin{aligned}\frac{d\mathbf{r}}{dt}&=-(m\sin mt)\mathbf{a}+(m\cos mt)\mathbf{b},\\ \frac{d^2\mathbf{r}}{dt^2}&=-(m^2\cos mt)\mathbf{a}-(m^2\sin mt)\mathbf{b}\\ &=-m^2\mathbf{r},\end{aligned}$$

and so
$$\frac{d^2\mathbf{r}}{dt^2}+m^2\mathbf{r}=\mathbf{0}.$$

8.



The direction of the line is parallel to **d**.

The velocity of P is $\dfrac{d\mathbf{r}}{dt}$, i.e.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = m\mathbf{d} \quad \text{since } m, \mathbf{a} \text{ and } \mathbf{d} \text{ are constant,}$$

so that $\mathbf{v}$ is a scaling of $\mathbf{d}$, i.e. the velocity vector is parallel to $\mathbf{d}$.

9.

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e} + r\frac{d\mathbf{e}}{dt}, \text{ and so}$$

$$\mathbf{e}\cdot\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}(\mathbf{e}\cdot\mathbf{e}) + r\left(\mathbf{e}\cdot\frac{d\mathbf{e}}{dt}\right).$$

Since $\mathbf{e}$ is a unit vector, $\mathbf{e}\cdot\mathbf{e} = 1$ and $\mathbf{e}\cdot\dfrac{d\mathbf{e}}{dt} = 0$, and so

$$\frac{dr}{dt} = \mathbf{e}\cdot\frac{d\mathbf{r}}{dt}.$$

10.

$$\frac{d\mathbf{u}}{dt} = -(2\sin 2t)\mathbf{i} - 2t\mathbf{j} + (e^t\sin t + e^t\cos t)\mathbf{k}.$$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \cos 2t - t^3 + 2\sin^2 t\\ &= (1 - 2\sin^2 t) - t^3 + 2\sin^2 t\\ &= 1 - t^3,\end{aligned}$$

and so

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = -3t^2.$$

11.

The velocity of the particle is the vector $\dfrac{d\mathbf{r}}{dt}$, and we have

$$\frac{d\mathbf{r}}{dt} = -(2\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k},$$

so a unit vector in the direction of the velocity is

$$\frac{d\mathbf{r}}{dt}\Bigg/\left|\frac{d\mathbf{r}}{dt}\right| = \frac{-(2\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}}{\sqrt{4\sin^2 t + \cos^2 t + 1}}.$$

When $t = \pi/2$, this is

$$\frac{-2\mathbf{i} + \mathbf{k}}{\sqrt{5}}.$$

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